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TERMINAL STATE IN A PREDICTIVE CONTROLLER COST FUNCTION

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Abstract: In the paper, an impact of a terminal state in a cost function on predictive controller behaviour is discussed for case of a set point tracking task. Attention is paid to the form of the terminal state added to the cost function and its effect to stability and quality of a feedback control. A complete design procedure of the predictive controller based on the state space description of controlled system is shown. The controller design includes the terminal state in a form of deviation from a desired terminal state. The concept of the desired terminal state opens the way to involve additional demands into the controller design. The stabilization effect of the terminal state in case of short control horizon is demonstrated on simulated control examples of a non-minimum phase system (system with unstable zero). The effect of the terminal state on the control quality is discussed, too.

Keywords: state space model, predictive control, terminal state, control stability and quality.

1 INTRODUCTION

A minimization of a quadratic cost function is common method for solving many engineering problems. In the control area this method is fundamental not only for some standard control design methods (LQ, predictive control) but also for a state estimation (Kalman estimator). Under the assumptions of linear controlled system and quadratic cost function it is possible to formulate the task of the controller design as a standard mathematical problem – extreme finding with an analytic solution. A unique solution exists also in the case of constrains existence in the form of linear inequalities.

The fundamental of the controller design is to incorporate maximum of known information and demands into properly formulated cost function. Because of the cost function minimization it is possible to involve various (even conflicting) control demands. Controller tuning consists in weightings of the particular demands.

From practical point of view is appropriate to formulate the task in discrete-time area with receding (finite) control horizon. The length N of the horizon is a parameter in the control design. The general formulation of a set point tracking task is given by Eq. (1a) – a state space description of controlled linear system dynamic behaviour with state and input variables constrains and by Eq. (1b) – a quadratic cost function J (control objective) with three terms. The cost function J depends on the horizon length N, the initial state $\mathbf{x}(k)$ (initial conditions in time k) and the time course of the future set point $\mathbf{w}_N$ (vector along the control horizon). Solution consists in computation of such a vector of system inputs $\mathbf{u}_N$, which leads to the minimum of the cost function and simultaneously respects all constrains.

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \qquad \mathbf{H}\mathbf{x}(k) \le \mathbf{h} \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \qquad \mathbf{G}\mathbf{u}(k) \le \mathbf{g} \end{aligned} \tag{1a}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are parameters of a discrete-time dynamic process model and
$\mathbf{H}$, $\mathbf{h}$, $\mathbf{G}$, $\mathbf{g}$ are parameters of state and input variables constrains.

$$\min_{\mathbf{u}_N} \left\{ \begin{aligned} J\big(N, \mathbf{u}_N, \mathbf{x}(k), \mathbf{w}_N\big) = \underbrace{\mathbf{e}_N^T \mathbf{Q} \mathbf{e}_N}_{J_e} + \underbrace{\mathbf{u}_N^T \mathbf{R} \mathbf{u}_N}_{J_u} + \\ + \underbrace{\mathbf{x}^T(k+N) \mathbf{Q}_x \mathbf{x}(k+N)}_{J_x} \end{aligned} \right\} \tag{1b}$$

$\mathbf{u}_N^T$=[$\mathbf{u}^T(k)$, $\mathbf{u}^T(k+1)$, … , $\mathbf{u}^T(k+N)$]
$\mathbf{y}_N^T$=[$\mathbf{y}^T(k)$, $\mathbf{y}^T(k+1)$, … , $\mathbf{y}^T(k+N)$]
$\mathbf{w}_N^T$=[$\mathbf{w}^T(k)$, $\mathbf{w}^T(k+1)$, … , $\mathbf{w}^T(k+N)$]
$\mathbf{e}_N = \mathbf{w}_N - \mathbf{y}_N$

where $\mathbf{Q}_x$, $\mathbf{Q}$, $\mathbf{R}$ are weighting matrices of particular terms.

The cost function always contains the fundamental control requirement (the term J_e) – the controlled outputs $\mathbf{y}$ of the system should reach the set point $\mathbf{w}$ (or follow its time course). This basic requirement is usually supplemented by another term J_u of the cost function. The term J_u implies the control costs. Thus the set point tracking is desired but not at the cost of arbitrarily large control actions.

The term J_x in the cost function (1b) can be used only in case of finite control horizon and state space description and it isn't common. It introduces into the cost function a dependence on the system state at the end of the control horizon – a terminal state. The predictive controller design based on input output description doesn't use it in a basic formulation of the cost function [Clarke 87a], [Clarke 87b], [Rossiter 03], [Camacho 07]. The terminal state is obviously introduced in some form only in extensions concerning to stability and robustness. Using of the terminal state increases in principle the stability of the controllers with finite horizon [Maciejowski 02]. The terminal state brings into the cost function dependence on all state variables. The cost function without terminal state depends only on the system outputs (or control error) and it can be independent from some state variables (which is given by matrix $\mathbf{C}$) and thus some states can increase ad infinitum even if the cost function is finite. In the case of control design based on input output models, where state doesn't exist in a nature form, the terminal state is replaced with a sequence of input and output variables. That approach of the terminal state treatment is called in the literature [Camacho 07], [Maciejowski 02] as a "terminal constrains".

In some cases the terminal state is important from mathematical point of view. In case of LQ control design on finite horizon, the mathematical importance of the terminal state is in that the matrix $\mathbf{Q}_x$ determines the initial value of a working matrix which is developing by iterating solution of discrete Riccati equation [e.g. Havlena 00].

In common literature about predictive control the terminal state is obviously mentioned only in context of controller stability. The using of terminal state has also an implication to the controller performance. The standard using of the terminal state in a form of eq. (1b) leads to permanent steady state control error in case of nonzero set point. This problem is easily solved by terminal state in a form of the deviation from a desired terminal state $\mathbf{x}_w$. The desired terminal state is a function of the set point and/or others demands. The one of the additional demands can be to achieve steady state at the end of control horizon. Additional optimization in terminal state can be an integral part of the controller due to the desired terminal state. Under the definition "optimization in terminal state" we will understand that controller ensures minimum of the weighted quadratic norm of a vector of deviations between desired and calculated terminal state.

Clear and unique requirements are possible to formulate because the state vector contains complete information about state of the system. For example if we have a system with more inputs than outputs there is an infinite number of input variables combinations that lead to the one output variables combination. But each inputs combination belongs to unique state. This idea is applied by a predictive control of a system with more inputs than outputs together with the demand of minimum energy cost [Dušek 07]. Set point temperature tracking of an ideal thermostatic bath actuated by the heating power and by the temperature of the inlet cooling water is deal with. In the literature [Dušek 08a], [Dušek 08b] problem how to determine an optimal steady state in the case of the systems with more inputs than outputs is discussed.

2 GENERAL PREDICTIVE CONTROLLER

The controller design starts from a discrete-time state space model of the controlled MIMO (Multi-Input Multi-Output) system with n_u inputs, n_x states and n_y outputs. The model is in a standard form (2a) – we suppose $\mathbf{D} = \mathbf{0}$ in the following text.

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k)\\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k)\end{aligned} \tag{2a}$$

where $\mathbf{u}(k)$ is vector of inputs with size $[n_u,1]$,
$\mathbf{x}(k)$ is state vector with size $[n_x,1]$ and
$\mathbf{y}(k)$ is the vector of outputs with size $[n_y,1]$.

Matrix equations (2b) describe vector of predicted system outputs $\mathbf{y}_N$ on the control horizon of length N. Vectors $\mathbf{y}_N$ and terminal state $\mathbf{x}(k+N)$ depend on the actual state $\mathbf{x}(k)$ and on a vector of future inputs $\mathbf{u}_N$.

$$\begin{aligned}\mathbf{x}(k+N) &= \mathbf{S}_{xx}\mathbf{x}(k) + \mathbf{S}_{xu}\mathbf{u}_N\\ \mathbf{y}_N &= \mathbf{S}_{yx}\mathbf{x}(k) + \mathbf{S}_{yu}\mathbf{u}_N\end{aligned} \tag{2b}$$

$\mathbf{u}_{N1}^T$=[$\mathbf{u}^T(k)$, $\mathbf{u}^T(k+1)$, … , $\mathbf{u}^T(k+N\text{-}1)$]
$\mathbf{y}_N^T$ = [$\mathbf{y}^T(k+1)$, … , $\mathbf{y}^T(k+N)$]

Matrices $\mathbf{S}_{xx}$, $\mathbf{S}_{xu}$, $\mathbf{S}_{yu}$ a $\mathbf{S}_{yu}$ depend on state space model parameters according to (2c).

With respect to a terminal state in the cost function (3a) in time instant $k+N$, the input vector has to be of length N and thus the vector is marked as $\mathbf{u}_N$.

$$\mathbf{S}_{xx} = \mathbf{A}^{N} \qquad \mathbf{S}_{xu} = \begin{bmatrix} \mathbf{A}^{N-1}\mathbf{B} & \mathbf{A}^{N-2}\mathbf{B} & \cdots & \mathbf{A}^{2}\mathbf{B} & \mathbf{A}^{1}\mathbf{B} & \mathbf{B} \end{bmatrix}$$

$$\mathbf{S}_{yx} = \begin{bmatrix} \mathbf{CA} \\ \mathbf{CA}^{2} \\ \vdots \\ \mathbf{CA}^{N-2} \\ \mathbf{CA}^{N-1} \end{bmatrix} \qquad \mathbf{S}_{yu} = \begin{bmatrix} \mathbf{CB} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{CA}^{1}\mathbf{B} & \mathbf{CB} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{CA}^{N-2}\mathbf{B} & \mathbf{CA}^{N-3}\mathbf{B} & \cdots & \mathbf{CAB} & \mathbf{CB} & \mathbf{0} \\ \mathbf{CA}^{N-1}\mathbf{B} & \mathbf{CA}^{N-2}\mathbf{B} & \cdots & \mathbf{CA}^{2}\mathbf{B} & \mathbf{CAB} & \mathbf{CB} \end{bmatrix} \tag{2c}$$

The cost function in matrix form (3a) changes from (1b) in a way of the terminal state application as a deviation from a desired terminal state $\mathbf{x}_W$ and that the vector of manipulated variable $\Delta\mathbf{u}_N$ is calculated as a deviations from a supposed future inputs $\mathbf{u}_{0,N}$.

$$J(N, \mathbf{u}_{N1}, \mathbf{x}(k), \mathbf{w}_N) = \mathbf{e}_N^T \mathbf{Q} \mathbf{e}_N + \Delta\mathbf{u}_N^T \mathbf{R} \Delta\mathbf{u}_N + \\ + \Delta\mathbf{x}^T(k+N)\mathbf{Q}_x \Delta\mathbf{x}(k+N) \tag{3a}$$

$$\Delta\mathbf{x}(k+N) = \mathbf{x}_W - \mathbf{x}(k+N)$$
$$\mathbf{e}_N = \mathbf{w}_N - \mathbf{y}_N$$
$$\mathbf{w}_N^T = [\mathbf{w}^T(k+1), \dots, \mathbf{w}^T(k+N)]$$
$$\Delta\mathbf{u}_N = \mathbf{u}_N - \mathbf{u}_{0,N}$$

where N is length of control horizon,
$\mathbf{x}_W$ is desired terminal state,
$\mathbf{w}_N$ is vector of future set points with size $[N \times n_y, 1]$,
$\mathbf{u}_{0,N}$ is vector of supposed future inputs with size $[N \times n_u, 1]$,
$\mathbf{u}_N$ is vector of optimal future inputs with size $[N \times n_u, 1]$,
$\mathbf{Q}_x$ is terminal state $\Delta\mathbf{x}$ weighting matrix with size $[n_x, n_x]$,
$\mathbf{Q}$ is control error $\mathbf{e}_N$ weighting matrix with size $[N \times n_y, N \times n_y]$ and
$\mathbf{R}$ is manipulated variable $\Delta\mathbf{u}_N$ weighting matrix with size $[N \times n_u, N \times n_u]$.

First item of vector $\mathbf{u}_N$ is applied as a manipulated variable $\mathbf{u}(k)$ every time instant and whole procedure is repeated. As the supposed future inputs (vector $\mathbf{u}_{0,N}$) constant vector filled with values of $\mathbf{u}(k-1)$ is considered and used in the following simulations.

2.1 *Desired terminal state*

Computation of desired terminal state is trivial in case of system with identical number of inputs and outputs and if we consider steady state. The solution for desired output $\mathbf{y}_0 = \mathbf{w}_0$ is

$$\mathbf{x}_0 = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\left[\mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right]^{-1}\mathbf{w}_0 \tag{4a}$$

Solution for a desired slope of outputs in terminal state is slightly more complicated. The solution for desired output $\Delta\mathbf{y}_0 = \Delta\mathbf{w}_0 = \mathbf{w}(N) - \mathbf{w}(N-1)$ is

$$\begin{aligned} \Delta\mathbf{x} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\left[\mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right]^{-1}[\mathbf{w}(N) - \mathbf{w}(N-1)] \\ \mathbf{u} &= \left[\mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\right]^{-1}\left[\mathbf{w}(N-1) + \mathbf{C}(\mathbf{I} - \mathbf{A})^{-1}\Delta\mathbf{x}\right] \\ \mathbf{x}_0 &= (\mathbf{I} - \mathbf{A})^{-1}(\mathbf{B}\mathbf{u} - \Delta\mathbf{x}) + \Delta\mathbf{x} \end{aligned} \tag{4b}$$

3 CONTROL EXPERIMENTS

The aim of the following control simulations is to demonstrate the effect of a terminal state in predictive controller design to control quality and stability. The simulations are supposed as an ideal case – controlled system is identical with the process model used for the controller design and neither noises nor disturbances are considered. The controller is designed for the set point tracking task.

Two different controlled systems are treated in simulations. The first system is a standard system of a higher order (5a) and the second one is a system with non stable zero (5b) i. e. with non-minimum phase. Both systems have similar settling time (circa 50 s). The step and impulse responses of both systems are in Fig. 1

$$F(s) = \frac{(9s+1)(9s+1)}{(5s+1)^5} \tag{5a}$$

$$F(s) = \frac{(9s+1)(-9s+1)}{(5s+1)^5} \tag{5b}$$

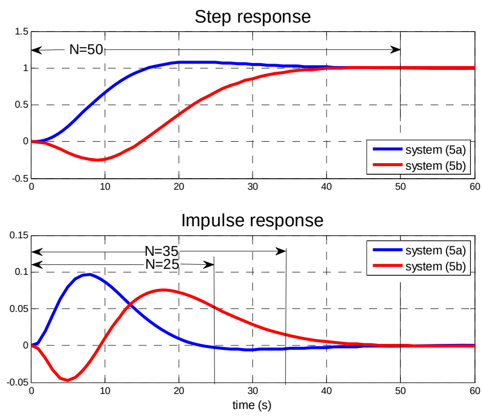

Figure 1 Characteristics of controlled systems

The input-output models (5a) and (5b) were converted into state space observable-canonical form. Predictive controller is described in Chapter 2. It operates in a nature way without modification with MIMO or SISO systems. The sampling time and control period is T = 1 s. The weighting matrices $\mathbf{Q}_x$, $\mathbf{Q}$ and $\mathbf{R}$ are diagonal. All diagonal elements of the weighting matrices are constant and their values are chosen so that the weight of every term in the cost function is comparable. From this reason weighting matrices are computed as reciprocal quadratic norms of corresponding steady state vectors according to (6). Tuning parameters of the controller are relative weightings ω and ω_x.

$$\mathbf{Q}_x = \omega_x \frac{\mathbf{I}}{\mathbf{x}_0^T \mathbf{x}_0} \qquad \mathbf{Q} = \frac{\mathbf{I}}{\mathbf{y}_{0N}^T \mathbf{y}_{0N}} \qquad \mathbf{R} = \omega \frac{\mathbf{I}}{\mathbf{u}_{0N}^T \mathbf{u}_{0N}} \tag{6}$$

where $\mathbf{x}_0$ is steady state vector,
$\mathbf{y}_{0N}$ is vector of constant outputs,
$\mathbf{u}_{0N}$ is vector of constant inputs,
ω_x is relative weight of matrix $\mathbf{Q}_x$,

ω is relative weight of matrix **R** and
I is identity matrix.

The set point shape consists from tree parts. The first part takes time as a control horizon plus 5 sampling times and the set point is constant. The second part lasts as a system settling time (50 s) and the set point linearly increases from the first to third part. The third part is as long as the second one and the set point is constant again. The control quality measure is calculated as integral of squared control error

$$ISE = T\sum_{k=1}^{NS} e^2(k) \quad (7)$$

where NS is number of samples during the control experiment.

The ISE criterion was chosen as the objective for control quality comparison. Discussed effect of terminal state can be observed from values of ISE measure for simulated control experiments summarized in Tab. 1 for system (5a) and in Tab. 2 for system (5b). The control experiments were simulated for several values of control horizon length N and terminal state relative weights ω_x. In all cases the desired terminal state is based on the demand of a desired slope of outputs in terminal state.

Table 1 ISE (×100) quality measure for system (5a)

T=1 s		ω =0.01		
N	ω_x =0.0	ω_x =0.1	ω_x =1.0	ω_x =10
2	unstable	unstable	**74.066**	**0.2953**
15	**0.6609**	**0.6773**	**0.6969**	**0.7009**
35	**0.6790**	**0.6785**	**0.6782**	**0.6782**
50	**0.6765**	**0.6764**	**0.6764**	**0.6763**

Table 2 ISE (×100) quality measure for system (5b)

T=1 s		ω =0.01		
N	ω_x =0.0	ω_x =0.1	ω_x =1.0	ω_x =10
15	unstable	unstable	unstable	**301.81**
20	unstable	**1063.6**	**258.62**	**51.685**
35	**21.965**	**15.778**	**8.8321**	**2.3089**
50	**2.0566**	**1.5504**	**1.0447**	**0.7062**

The control responses of two selected control experiments are shown in Fig. 2a – controlled system (5a) and in Fig. 2b – controlled system (5b). Both experiments are considered with identical parameters – the length of control horizon is N=35 and the relative gain of the terminal state is ω_x =0.1.

4 CONCLUSIONS

Effect of the terminal state to stability of the feedback control is definitely positive. Especially in the case of problematic system (5b) and wrong choice of the controller parameters (too short control horizon) the terminal state increases dramatically the controller stability.

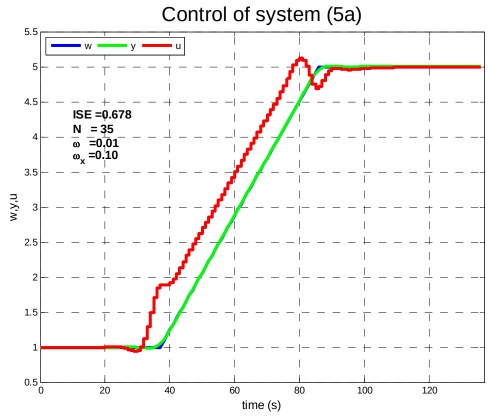

Figure 2a Control response of system (5a)

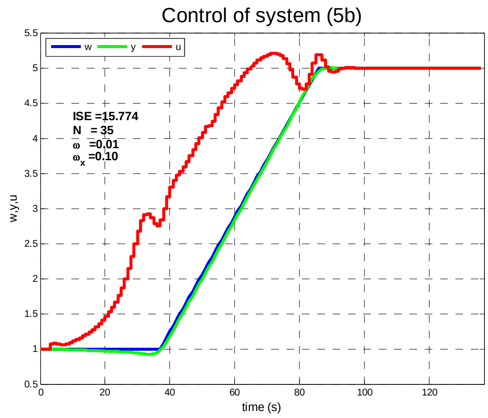

Figure 2b Control response of system (5b)

The control quality is better if the terminal state is used in a form of deviation from the desired terminal state (calculated for steady state or desired slope of outputs) and the state space observable-canonical form. If the state space controllable-canonical form is used, then the control quality of the standard system (5a) is worse. In this case the cost function minimization leads to a large initial items of the calculated vector $\Delta\mathbf{u}_N$ even if only one item of the set point vector $\mathbf{w}_N$ is changed (the last item $\mathbf{w}(k+N)$ at the end of the horizon). That situation is demonstrated in Fig. 3a in comparison with using of the observable-canonical form in Fig. 3b.

In both cases the desired terminal state is based on a demand of slope of outputs in terminal state corresponding to the slope of set point at the end of the control horizon.

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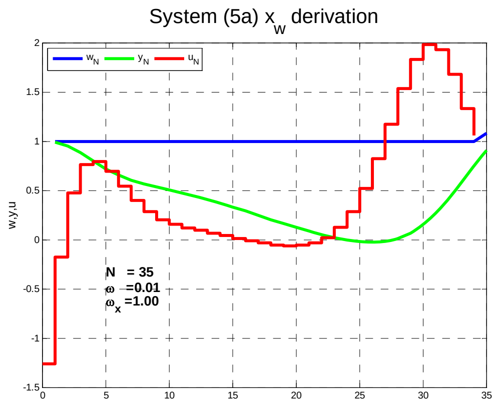

Figure 3a Prediction along control horizon controllable-canonical form

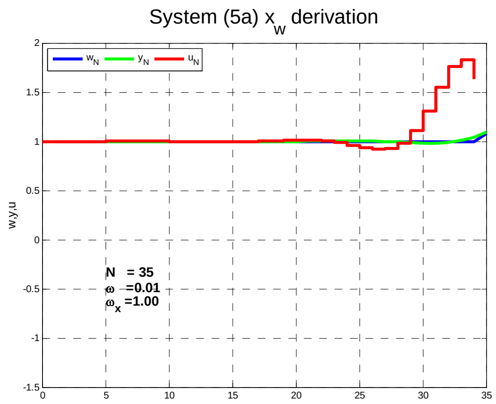

Figure 3 Prediction along control horizon observable-canonical form

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