Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

17^{th} International Conference on Process Control 2009

Hotel Baník, Štrbské Pleso, Slovakia, June 9 – 12, 2009

ISBN 978-80-227-3081-5

http://www.kirp.chtf.stuba.sk/pc09

Editors: M. Fikar and M. Kvasnica

Huspeka, J.: Second Order Sliding Mode Control of the DC Motor, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 17th International Conference on Process Control '09*, Štrbské Pleso, Slovakia, 134–139, 2009.

Full paper online: http://www.kirp.chtf.stuba.sk/pc09/data/abstracts/051.html

SECOND ORDER SLIDING MODE CONTROL OF THE DC MOTOR

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Abstract: This paper deals with the second order sliding mode control algorithm known as terminal sliding mode control. The main advantage of this control method lies in the finite-time convergence of the switching variable and its first derivation to zero. Firstly, the paper introduces the principle of sliding mode control method. The main result of the paper is the analysis the terminal sliding mode control. Then, the comparison between first and second sliding mode control is made on the real model of the DC motor. At last, simulation results are presented.

Keywords: Sliding mode, control, higher order sliding mode, DC motor

1. INTRODUCTION

The sliding mode (SM) control, known also as the variable structure control, is nonlinear method of the feedback control. SM control is realized by switching of the feedback discontinuous in time between at least two smooth functions. Therefore, the structure of the control law changes due to the location of the state trajectory in the state space. The most common SM control method is the relay in the feedback. This relay switches according some switching function. The structure of the switching function is designed in order to attract the trajectory in the state space to the switching surface. This is the manifold, where switching function equals zero. The part of the state space, where the state trajectory slides along this switching surface is called sliding mode. There are two main advantages of this control: robustness and finite time convergence. The basic SM control algorithm provides finite time convergence of the switching function to the switching surface and the control appears in the first derivation of the switching function. This brings the disadvantage of SM control algorithm called chattering, which is behavior of the trajectory in the vicinity of the sliding mode in presence of the switching imperfection, eg. relay with hysteresis. On the contrary, the second order sliding mode (2SM) control provides finite time convergence of the switching function and its first derivation to zero, see Levant (2007). Moreover, the control law firstly appears in the second derivation of the switching function. This brings the main benefit of 2SM control method: the first derivation of the switching function is piecewise smooth function, therefore the switching function is smooth. This behavior will significantly reduce the chattering and at the same time keep the advantages of SM control method, see Levant (1993). In this paper, SM control and 2SM control methods are compared by simulation of the DC motor control.

2. SLIDING MODE CONTROL

The main aim of the SMC method is to design the controller, which leads the sliding function to the manifold and keep the system in desired sliding mode, see T. Floquet (2002). The SMC design consists of two steps: switching function design and controller design. The switching function is the function of the state and can be obtain either as output or state feedback. This section is focused on the latter method. The following procedure of the SMC design is introduced in Monsees (2002). Consider a linear system

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$. The matrix B is assumed to have full rank and (A, B) is controllable. The control law u is defined

$$u = -K\mathrm{sgn}\sigma \tag{2}$$

where K is positive constant and σ is the switching function.

2.1 *Switching function*

The design of the switching function consists in finding of the matrix $S \in \mathbb{R}^n$, which defines switching function

$$\sigma = Sx \tag{3}$$

which provides desired dynamics in sliding mode. The first step is to find a transformation matrix $T \in \mathbb{R}^{n\times n}$

$$\bar{x} = Tx \tag{4}$$

which transforms the system (1) into so called regular form, see Monsees (2002)

$$\begin{aligned}\dot{\overline{x}}_1 &= \overline{A}_{11}\overline{x}_1 + \overline{A}_{12}\overline{x}_2 \\ \dot{\overline{x}}_2 &= \overline{A}_{21}\overline{x}_1 + \overline{A}_{22}\overline{x}_2 + \overline{B}_2 u\end{aligned} \tag{5}$$

with $\overline{x}_1 \in \mathbb{R}^{n-1}$, $\overline{x}_2 \in \mathbb{R}$, $\overline{A}_{11} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\overline{A}_{12} \in \mathbb{R}^{n-1}$, $\overline{A}_{21} \in \mathbb{R}^{n-1}$, $\overline{A}_{22} \in \mathbb{R}$ and $\overline{B}_2 \in \mathbb{R}$. The switching function (3) is transformed in the new coordinates

$$\sigma = \overline{S}_1\overline{x}_1 + \overline{S}_2\overline{x}_2 \tag{6}$$

where $\overline{S}_1 \in \mathbb{R}^{n-1}$, $\overline{S}_2 \in \mathbb{R}$. These matrices are design parameters defining the sliding surface $\sigma = 0$. Assuming a controller, which brings the system in the sliding mode, the state variable $\bar{x}_2$ is computed from (6)

$$\overline{x}_2 = -\overline{S}_2^{-1}\overline{S}_1\overline{x}_1 \tag{7}$$

The matrix $\overline{S}_2$ is assumed invertible. Substituting (7) into (5) results in dynamics in sliding mode

$$\dot{\overline{x}}_1 = \left(\overline{A}_{11} - \overline{A}_{12}\overline{S}_2^{-1}\overline{S}_1\right)\overline{x}_1 \tag{8}$$

It is necessary to choose one matrix in the product $\overline{S}_2^{-1}\overline{S}_1$. The common choice is

$$\overline{S}_2 = \overline{B}_2^{-1} \tag{9}$$

which ensures matrix $\bar{S}_2$ is invertible. The eigenvalues of the matrix $\left(\overline{A}_{11} - \overline{A}_{12}\overline{S}_2^{-1}\overline{S}_1\right)$ can be assigned by some standart state feedback method eg. pole placement and matrix $\bar{S}_1$ will be computed. The matrix S in (3) is then obtained from

$$S = [\,\overline{S}_1\ \overline{S}_2\,]T \tag{10}$$

2.2 *Control law design*

In order to find a controller ensuring sliding mode of the system (5) occurs in finite time, following transformation matrix is defined

$$\begin{bmatrix}\bar{x}_1 \\ \sigma\end{bmatrix} = \begin{bmatrix} I & 0 \\ \overline{S}_1 & \overline{S}_2\end{bmatrix}\bar{x} \tag{11}$$

Applying transformation (11) on the system (5) brings the system in form

$$\dot{\overline{x}}_1 = \tilde{A}_{11}\overline{x}_1 + \tilde{A}_{12}\sigma \tag{12}$$

$$\dot{\sigma} = \tilde{A}_{21}\overline{x}_1 + \tilde{A}_{22}\sigma + u \tag{13}$$

where all matrices have appropriate dimensions. The control law is chosen

$$u = u_c + u_d \tag{14}$$

The continuous component u_c and discontinuous component u_d are given by

$$u_c = -\tilde{A}_{21}\overline{x}_1 - \tilde{A}_{22}\sigma \tag{15}$$

$$u_d = -K_s\mathrm{sgn}\sigma - K_p\sigma \tag{16}$$

To study a stability the Lyapunov function is defined

$$V(\sigma) = \frac{1}{2}\sigma^2 \tag{17}$$

To this function can by applied reaching law

$$\dot{V}(\sigma) = \sigma \cdot \dot{\sigma} \le -\eta|\sigma| \tag{18}$$

with positive constant η. Substituting (13)-(16) into (18) results in

$$-K_s|\sigma| - K_p\sigma^2 \le -\eta|\sigma| \tag{19}$$

Dividing (19) by $|\sigma|$, the condition (19) leads to

$$K_s + K_p|\sigma| \ge \eta \tag{20}$$

This condition is always satisfied when $K_s \ge \eta$ and $K_p \ge 0$, which is used for accelerating the convergence of the trajectory to the switching surface. If these conditions are met, the closed loop system will reach the sliding mode in finite time.

3. TERMINAL SLIDING MODE

The sliding mode control of the higher order (HOSM) is generalisation of the sliding mode control shown in previous section. There are additional conditions on the sliding variable to be satisfied in HOSM. These conditions can be defined as the intersection of the manifolds

$$\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0 \tag{21}$$

where r is the order of the sliding mode. The order of the sliding mode is the r-th time derivation of the sliding variable, where the control u firstly appears.

$$\frac{\partial\sigma^{(r)}}{\partial u} \ne 0 \tag{22}$$

The aim of the HOSM control design is to find the control law in order that the state trajectory reaches the intersection (21) in finite time. In this section, the 2SM controller known as terminal sliding mode or the sliding mode controller with prescribed convergence law will be analyzed. Consider the linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_1 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} u \quad (23)$$

where $a_i, b > 0$ and the terminal sliding mode control law

$$u = -K \mathrm{sgn}(\underbrace{\dot{\sigma} + \lambda\sqrt{|\sigma|}\mathrm{sgn}\sigma}_{=\rho}) \quad (24)$$

with $K, \lambda > 0$. This control law is analysed as the SM control method with switching variable ρ.

3.1 *Switching variable*

Suppose, that the sliding mode on the manifold $\rho = 0$ occures. Following equations hold for switching variable ρ on the manifold.

$$\rho = \dot{\sigma} + \lambda\sqrt{|\sigma|}\mathrm{sgn}(\sigma) = 0 \quad (25)$$

$$\dot{\rho} = \ddot{\sigma} + \lambda \frac{\partial\left(\sqrt{|\sigma|}\mathrm{sgn}(\sigma)\right)}{\partial t} = 0 \quad (26)$$

The equation (25) is the switching surface and the equation (26) is the condition that the state of the system does not leave the manifold. Moreover, from (26) can be computed the continuous control, which will keep the state on the manifold, so called equivalent control. In order to solve these equations, the switching variable σ has to be designed. From (22) results that σ must not be the function of the third state component x_3 in order that control u has to appear firstly in the second derivation of σ. This condition will be satisfied for

$$\sigma = s_1 x_1 + s_2 x_2 \quad (27)$$

The equation (26) has to be analyzed in three steps considering the sign of the switching function σ.

a) For $\sigma > 0$ results (26) in

$$\dot{\rho} = \ddot{\sigma} + \frac{\lambda\dot{\sigma}}{2\sqrt{\sigma}} = 0 \quad (28)$$

In (28) will be substituted (23),(27)

$$s_1 x_3 + s_2 \dot{x}_3 + \frac{\lambda(s_1 x_2 + s_2 x_3)}{2\sqrt{s_1 x_1 + s_2 x_2}} = 0 \quad (29)$$

From (29) will be computed equivalent control u_{eq}

$$u_{eq} = -\frac{s_1}{b s_2} x_3 - \frac{\lambda\,(s_1 x_2 + s_2 x_3)}{2 b s_2 \sqrt{s_1 x_1 + s_2 x_2}} + \frac{1}{b}\sum_{i=1}^{3} a_i x_i \quad (30)$$

This equivalent control is substituted in (23) and the so called equivalent dynamics in sliding mode on the manifold (25) results

$$\dot{x}_{1,eq} = x_{2,eq} \quad (31)$$

$$\dot{x}_{2,eq} = -\frac{s_1}{s_2} x_{2,eq} - \frac{\lambda}{s_2}\sqrt{s_1 x_{1,eq} + s_2 x_{2,eq}} \quad (32)$$

b) For $\sigma < 0$ results (26) in

$$\dot{\rho} = \ddot{\sigma} + \frac{\lambda\dot{\sigma}}{2\sqrt{-\sigma}} = 0 \quad (33)$$

In (33) will be substituted (23),(27)

$$s_1 x_3 + s_2 \dot{x}_3 + \frac{\lambda(s_1 x_2 + s_2 x_3)}{2\sqrt{-(s_1 x_1 + s_2 x_2)}} = 0 \quad (34)$$

From (34) will be computed equivalent control u_{eq}

$$u_{eq} = -\frac{s_1}{b s_2} x_3 - \frac{\lambda\,(s_1 x_2 + s_2 x_3)}{2 b s_2 \sqrt{-(s_1 x_1 + s_2 x_2)}} + \frac{1}{b}\sum_{i=1}^{3} a_i x_i \quad (35)$$

This equivalent control is substituted in (23) and the so called equivalent dynamics in sliding mode on the manifold (25) results

$$\dot{x}_{1,eq} = x_{2,eq} \quad (36)$$

$$\dot{x}_{2,eq} = -\frac{s_1}{s_2} x_{2,eq} + \frac{\lambda}{s_2}\sqrt{-(s_1 x_{1,eq} + s_2 x_{2,eq})} \quad (37)$$

c) For $\sigma = 0$ results (26) in

$$\dot{\rho} = \ddot{\sigma} = 0 \quad (38)$$

In (38) will be substituted (23),(27) and the equivalent control is computed

$$u_{eq} = -\frac{s_1}{b s_2} x_3 + \frac{1}{b}\sum_{i=1}^{3} a_i x_i \quad (39)$$

This equivalent control is substituted in (23) and the so called equivalent dynamics in sliding mode on the manifold (25) results

$$\dot{x}_{1,eq} = -\frac{s_1}{s_2} x_{1,eq} \quad (40)$$

This equivalent dynamics describes the behavior of the system (23) in 2SM, because the state of the system fulfills the condition (21).

The parameters s_1, s_2 has to be designed to reach the desired behavior in each part of the manifold (25), eg. pole placement of the equivalent dynamics (40). Possible choice of the parameter s_2 is

$$s_2 = \frac{1}{b} \quad (41)$$

which ensures s_2 to be invertible.

3.2 *Control law*

The control law of the 2SM control is composed of the continuous and discontinuous part

$$u_{2SM} = u_{c,2SM} + u_{d,2SM} \quad (42)$$

The discontinuous part $u_{d,2SM}$ is the control law (24) and the continuous part $u_{c,2SM}$ is chosen as equivalent control (39)

$$u_{c,2SM} = -\frac{s_1}{bs_2}x_3 + \frac{1}{b}\sum_{i=1}^{3} a_i x_i \tag{43}$$

$$u_{d,2SM} = -K\mathrm{sgn}(\dot{\sigma} + \lambda\sqrt{|\sigma|}\mathrm{sgn}\sigma) \tag{44}$$

To study a stability the Lyapunov function is defined

$$V(\rho) = \frac{1}{2}\rho^2 \tag{45}$$

To this function can by applied reaching condition

$$\dot{V}(\rho) = \rho \cdot \dot{\rho} < 0 \tag{46}$$

Substituting (25), (26) into (46) leads to the condition for design parameters K, λ in (44).

$$\frac{\lambda^2}{2} < s_2 b K \tag{47}$$

If (47) holds, the switching variable ρ reaches its manifold in finite time. Moreover, the state slides along this manifold and reaches the second order sliding mode of the switching variable σ also in finite time.

4. DC MOTOR CONTROL

The DC motor is the actuator of many industrial processes. The mathematical model of this motor is described by following representation.

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{k_m}{J} \\ 0 & -\frac{k_e}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u \tag{48}$$

where state variable x_1 is an angle of the shaft, x_2 is the angular velocity of the shaft and x_3 is the current of the armature coil. Input u is the voltage of the armature coil. The parameters R, L are inductance and resistance of the armature coil, k_e is the speed constant, b is the viscous friction, J is the moment of inertia and k_m is the torque constant.

4.1 *SM control design*

The state speace representation of the DC motor (48) is already in the form (5). This means the transformation matrix (4) equals

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{49}$$

The switching function (3) for the control of the DC motor (48) in new coordinates is

$$\sigma = \bar{s}_1\bar{x}_1 + \bar{s}_2\bar{x}_2 + \bar{s}_3\bar{x}_3 \tag{50}$$

In the sliding mode, the state variable $\bar{x}_3$ is computed from (6)

$$\bar{x}_3 = -\bar{s}_3^{-1}(\bar{s}_1\bar{x}_1 + \bar{s}_2\bar{x}_2) \tag{51}$$

Parameter $\bar{s}_3$ is chosen according to (9)

$$\bar{s}_3 = \bar{B}_2^{-1} = L \tag{52}$$

Substituting (51), (52) into (48) results in dynamics in sliding mode

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_m}{L_k J}\bar{s}_{11} & -\frac{b + k_m}{L_k J}\bar{s}_{12} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{53}$$

The parameters of the real motor substitute in (53): $L_k = 0,953 \cdot 10^{-3}H$, $R_k = 7,17\Omega$, $k_e = 0.29Vs$, $k_m = 46 \cdot 10^{-3}Nm \cdot A^{-1}$, $J = 4,42 \cdot 10^{-6}kg \cdot m^2$, $b = 2,99 \cdot 10^{-4}Nm \cdot s$. The description of this motor can be find in Maxon (2009).

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1,09 \cdot 10^7\bar{s}_1 & -1,10 \cdot 10^7\bar{s}_2 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \tag{54}$$

The eigenvalues of the matrix of the dynamics in (54) will be assigned $\lambda_{1,2} = -100$ by pole placement method. The coefficients $\bar{s}_1, \bar{s}_2$ are computed

$$\bar{s}_1 = 9.16 \cdot 10^{-4} \tag{55}$$

$$\bar{s}_2 = 1,82 \cdot 10^{-5} \tag{56}$$

The switching function σ is obtained using (10), (49).

$$\sigma = 9.16 \cdot 10^{-4}x_1 + 1,82 \cdot 10^{-5}x_2 + 9,53 \cdot 10^{-4}x_3 \tag{57}$$

The controller, which brings the state trajectory to the manifold will have following structure

$$u = 0,2903x_2 + 6,9806x_3 - K_s\mathrm{sgn}\sigma - K_p\sigma \tag{58}$$

4.2 *Terminal sliding mode control design*

Firstly, the state representation (48) has to be transformed in form (23) by following transformation

$$x_f = Tx_{DC} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{k_m}{J} \end{bmatrix} x_{DC} \tag{59}$$

where index f denotes Frobenius form and DC the original representation of the DC motor. The Frobenius form of the motor is

$$\dot{x}_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{k_m k_e + bR}{JL} & -\frac{JR + bL}{JL} \end{bmatrix} x_f + \begin{bmatrix} 0 \\ 0 \\ \frac{k_m}{JL} \end{bmatrix} u \tag{60}$$

According to (41), s_2 is chosen

$$s_2 = \frac{1}{b} = \frac{JL}{k_m} \tag{61}$$

The parameter s_1 is designed by pole placement method using dynamics in 2SM (40). From characteristic polynom of (40) and desired dynamics with pole $\lambda = 100$ follows

$$s_1 = 100 s_2 = \frac{100JL}{k_m} \tag{62}$$

The continuous part of the control law of the Frobenius form is computed using (43)

$$u_{c,f} = -\frac{100JL}{k_m} x_{f,3} + \frac{k_m k_e + bR}{k_m} x_{f,2} + \\ + \frac{JR + bL}{k_m} x_{f,3} \tag{63}$$

The parameters of the motor are substituted in (59), (60), (63) and the continuous control of the representation of the motor (48) is obtained by substituting (59)

$$u_c = 0,2902 x_2 + 7,139 x_3 \tag{64}$$

The switching function σ of the discontinuous part (44) and its derivation $\dot{\sigma}$ also have to be transformed

$$\sigma = s_1 x_{1,f} + s_2 x_{2,f} = s_1 x_1 + s_2 x_2 \tag{65}$$

$$\dot{\sigma} = s_1 x_{2,f} + s_2 x_{3,f} = s_1 x_2 - \frac{s_2 b}{J} x_2 + \frac{s_2 k_m}{J} x_3 \tag{66}$$

Parameter λ in (47) has to be chosen according the condition

$$\frac{\lambda^2}{2} < K \tag{67}$$

5. RESULT

The simulation experiment is realized for SM control with parameters $K_s = 1$, $K_p = 0$. The 2SM control is simulated three times with parameter $K = 1$ and different values of $\lambda = 0,6; 1; 1,4$, because the limit value for λ is $\sqrt{2K}$ according to (67). The result of this experiment is depicted on Fig. 1-3. The settling time of the responses of the 2SM control simulations is much lower even for low parameter λ. The choice of this parameter affects the amplitude of the current of the armature coil of the motor. It is clear, that this amplitude may be restricted in practical realization. This restriction will be fulfilled by appropriate choice of λ. This choice will shape the transfer functions of the simulation experiment using 2SM control, because this control firstly brings the trajectory in the first order sliding mode of the switching function ρ, where λ influences the dynamics. The second order sliding mode itself occurs when the trajectory reaches the intersection of the manifolds $\sigma = 0$ and $\dot{\sigma} = 0$.

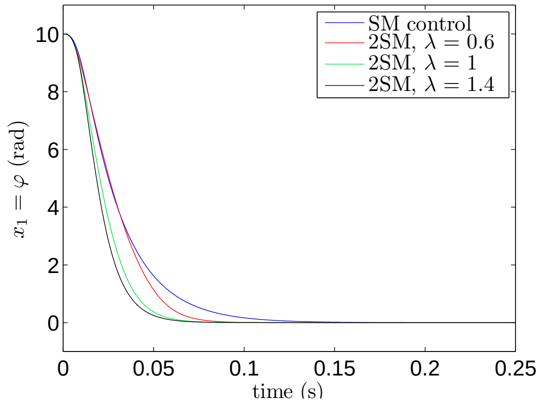

Fig. 1. The angular position of the shaft of the motor

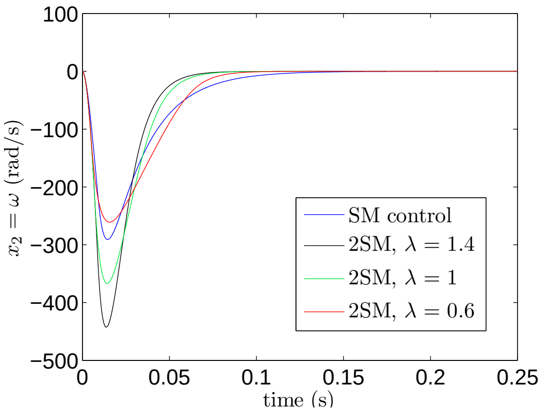

Fig. 2. The angular speed of the shaft of the motor

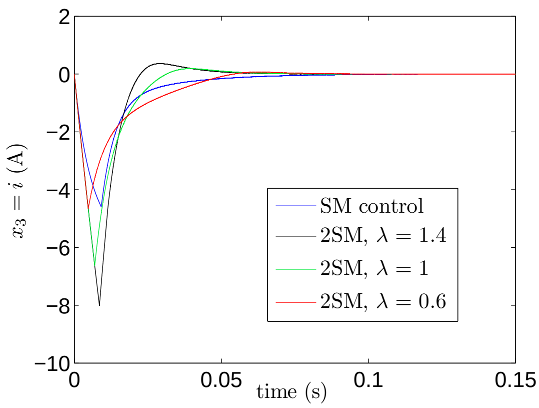

Fig. 3. The current of the armature coil of the motor

6. CONCLUSION

This paper presents the control design of the sliding mode of the first and second order. Both these methods demands the design of the sliding function with respect to the dynamics in sliding mode. The same choice of the poles of the dynamics in sliding mode is used in both methods of control. Secondly, there is used the same structure of the control law, which brings the system in the sliding mode in finite time. Moreover, the 2SM control algorithm brings the first derivation of the switching function to zero in finite time. The comparison of this two methods is presented by simulation of the DC motor control. It is shown, that the 2SM control algorithm has the optional design parameter, which allows to shape the transfer function.

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