Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

Editors: M. Fikar and M. Kvasnica

Vyskočil, M., Schlegel, M.: Improved Procedure for Robustness Regions Design, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 17th International Conference on Process Control '09*, Štrbské Pleso, Slovakia, 140–144, 2009.

Full paper online: `http://www.kirp.chtf.stuba.sk/pc09/data/abstracts/075.html`

IMPROVED PROCEDURE FOR ROBUSTNESS REGIONS DESIGN

Martin Vyskočil[*], M. Schlegel[**]

** Department of Cybernetics, University of West Bohemia in Pilsen
fax : +420377632502 and e-mail : m.vyskoc@seznam.cz*

*** Department of Cybernetics, University of West Bohemia in Pilsen
fax : +420377632502 and e-mail : schlegel@kky.zcu.cz*

Abstract: The design problem of robustness controllers for LTI systems is a fundamental problem in control theory. Some robustness region methods for PID controllers based on frequency design specifications (e.g. conditions on the maximum of sensitivity and complementary sensitivity functions) were published recently. However, these methods find all roots of a high order polynomial equation in each step and consequently are very demanding with respect to computation time. The paper introduces more effective procedure for determination of robustness regions based on the classical Laguerre's root finding algorithm.

Keywords: robustness design, parametric plane method

1 INTRODUCTION

The classic PI/PID controllers are still widely used in process industry. The main reason of their popularity is their simplicity and possibility to tune the controller without the process model.

There are several method of design PI/PID controllers suitable for practice. Most of them are semi-empirical and applicable only for some specific kind of controlled systems. There exist only a few general methods applicable for non rational transfer functions, e.g. time delay systems and fractional order systems. Most of them are based on parameter plane methods, e.g. D-partition method (Neimark 1948) or robustness regions method (Shafiei 1997, Schlegel 2003).

These methods are mainly based on open-loop shaping to obtain the proper maximum value of sensitivity and complementary sensitivity functions. For example, the restrictions are maximal values of sensitivity and complementary sensitivity functions in the defined frequency interval. These restrictions specify circle areas in the complex plane, where Nyquist plot have not to be placed. The design specifications for specific type of controller isolates the sensitivity and complementary sensitivity regions (called robustness regions) in the parameter plane. Intersection of all robustness regions determines set of controller parameters, which complies all design specifications.

In this paper we consider design requirements for maximum value of complementary sensitivity function. The open-loop Nyquist plot must not to have any intersection with given circle except one tangential point. This leads to solving high order polynomial equation in the each step. Previously published work (Schlegel 2003) finds always all roots in the each step which demands much computation time. To decreasing the computation time the Laguerre's root finding algorithm is used. There are computed only such roots, which are really needed.

2 CONTROL LOOP

Because of simplicity, we restrict ourselves only to the PI controller design only. The presented design procedure can be simply extended to others controller types, but the corresponding equations are more complex.

Consider feedback control loop shown in Fig. 1, where F_C, F_P are controller transfer function a process transfer function respectively. The PI controller is given by:

$$F_C(j\omega) = k + \frac{k_i}{j\omega} = k + \frac{kd}{j\omega}, \quad k, d > 0 \qquad (1)$$

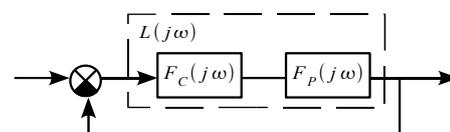where k, k_i are its real positive engineering parameters and $k_i = kd$.

Figure 1: The general feedback control system

The process transfer function is defined by

$$F_P(j\omega)=a(\omega)+j\,b(\omega) \qquad (2)$$

where $a(\omega)$, $b(\omega)$ are arbitrary real part and imaginary part functions with parameter ω.

Now, open-loop transfer function is given by

$$L(j\omega)=\underbrace{\left(a\,k+\frac{b\,k\,d}{\omega}\right)}_{u(\omega)}+j\underbrace{\left(b\,k-\frac{a\,k\,d}{\omega}\right)}_{v(\omega)} \qquad (3)$$

Further denote:

$$\frac{\partial}{\partial\omega}L(j\omega)=u_1(\omega,k,d)+j\,v_1(\omega,k,d) \qquad (4)$$

$$\frac{\partial}{\partial\omega}F_P(j\omega)=a_1(\omega)+j\,b_1(\omega) \qquad (5)$$

where u_1, v_1 are given by following equations

$$u_1=\frac{d\,k\,\omega\,b_1(\omega)+k\,\omega^2 a_1(\omega)-d\,k\,b(\omega)}{\omega^2} \qquad (6)$$

$$v_1=\frac{k\,\omega^2 b_1(\omega)-d\,k\,\omega\,a_1(\omega)+d\,k\,a(\omega)}{\omega^2} \qquad (7)$$

3 ROBUSTNESS REGIONS

Consider only the design requirements to maximum value of sensitivity $S(j\omega)$ and complementary $T(j\omega)$ sensitivity functions, defined by

$$S(j\omega)=\frac{1}{1+L(j\omega)}, \qquad (8)$$

$$T(j\omega)=\frac{L(j\omega)}{1+L(j\omega)} \qquad (9)$$

The maximum sensitivity M_S and maximum complementary sensitivity M_T are, respectively, defined by

$$M_S=\sup_{\omega}|S(j\omega)|, \quad M_T=\sup_{\omega}|T(j\omega)| \qquad (10)$$

Both of these constraints can be interpreted as a limitation on the open-loop $L(j\omega)$ in the following form: Nyquist plot $L(j\omega)$, $\omega\in\langle 0,\infty)$ is outside a circle with center c and radius $r>0$.

The sensitivity M_S corresponds with:

$$c=-1, \quad r=\frac{1}{M_S} \qquad (11)$$

and complementary sensitivity M_T with

$$c=-\frac{M_T^2}{M_T^2-1}, \quad r=\frac{M_T}{|M_T^2-1|} \qquad (12)$$

The Nyquist plot $L(j\omega)$ is tangential to the circle with the centre c and radius r at the frequency ω if it holds

$$(u-c)^2+v^2=r^2 \qquad (13)$$

and

$$(u-c)u_1+v\,v_1=0 \qquad (14)$$

where the equation (14) express the requirement for the Nyquist plot to be tangential to the circle considered. The situation is shown in the Fig. 2.
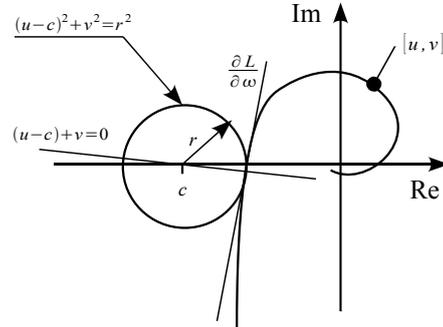

Figure 2. Circle robustness region

We want to find all points in the k-k_i parameter plane which satisfy above stated conditions (13, 14) for the Nyquist plot $L(j\omega)$.

Using (3, 6, 7), we obtain from (13) the following equation for controller parameter k

$$k=(a_1 c\,\omega^3+b_1 c\,d\,\omega^2-b\,c\,d\,\omega)/$$
$$((bb_1+aa_1)\omega^3+(bb_1+aa_1)d^2\omega+ \qquad (15)$$
$$+(-b^2-a^2)d^2)$$

By substituting (15) into (14), we obtain the following polynomial equation (16) for indeterminate variable d. It is 4th-degree order polynomial equation for here considered PI controller (but it is 7th-degree order for PID controller).

$$c_4 d^4+c_3 d^3+c_2 d^2+c_1 d+c_0=0 \qquad (16)$$

For brevity the coefficients c_i are given in the appendix.

Let $^l d$, $l=1,2,\dots,m$ denote the real positive roots of the polynomial equation (16), then the pairs $[\underbrace{^l k(^l d)}_{^l k},\underbrace{^l k(^l d)^l d}_{^l k_i}]$ describe parametric curves with parameter ω which creates the boundaries of robustness regions in the parameter plane $k-k_i$.

4 ENHANCED ALGORITHM

Finding all roots for every frequency ω in each step demands very much computation time. It is obvious that only some result roots are really needed. For example, for PI controller is suitable mostly only one root. The inconvenient roots are mostly negative, complex conjugate or designed close-loop system is unstable for them.

In this paper, the main idea is to compute only such roots of (16), which are really needed. Consider that

we have vector of pairs $\left[{}^l k \left({}^l d \left(\omega_j \right) \right), k_i \, {}^l \left(d^l \left(\omega_j \right) \right) \right]$ for given frequencies $\omega_j \in \Omega$, which creates the boundary of robustness region in the parameter plane $k - k_i$. Let holds $\omega_j \approx \omega_{j+1}$ for every $\omega_j \in \Omega$. From this follows that the result of (16) for ω_j, ω_{j+1} is almost same, i.e. ${}^l d \left(\omega_j \right) \approx {}^l d \left(\omega_{j+1} \right)$.

For finding roots of (16) the approximation Laguerre's root finding algorithm is used. More details about this zerofinding method are in (Wankere 2001). The result ${}^l d \left(\omega_j \right)$ in previous step is used as initial approximation for finding solution ${}^l d \left(\omega_{j+1} \right)$ in the next step.

The robustness regions algorithm can be described in the following steps:

(1) Choose logarithmic sequence $\Omega = \{ \omega_1, \omega_2, \ldots, \omega_N \}$
Set $j = 1$

(2) Find all roots of (16) for frequency $\omega = \omega_j$, choose real positive roots ${}^l d \left(\omega_j \right)$, $l = 1, 2, \ldots m$.

(3) By using Laguerre's method and the result from previous step as initial approximation the root ${}^l d \left(\omega_{j+1} \right) = f \left(d^l \left(\omega_j \right), \omega_j \right)$ is computed.

(4) If the found roots ${}^l d \left(\omega_{j+1} \right)$ are suitable go to the next step else return back to the step 2. If the second attempt is unsuccessful too, the solution for ω_{j+1} probably does not exist, go the step 5.

(5) Set $j = j + 1$. If holds $j < N$ go back to the step 3, else go to the step 6.

(6) The End – all pairs $k - k_i$ are computed

EXAMPLE – HEATING PROCESS

Assume thermal process - metal rod heated by Peltier heater, which is shown in Fig 3. The rod is heated at the one end and the temperature is measured at the second temperature isolated end. The length of the rod is l_r. The actuating variable *u(t)* is heat energy put into or taken out of the metal rod. The process (controlled) variable *y(t)* is temperature of the isolated end of the rod. The temperature energy can lost nowhere. The plant is described by following partial differential equation.

$$\frac{\partial x(t, \xi)}{\partial t} = \kappa \, \frac{x(t, \xi)}{\partial \xi^2} \qquad (17)$$

where $x(t, \xi)$ is temperature in the length of ξ at the time *t*.

Boundary condition of the heated end

$$\frac{\partial x(t, \xi)}{\xi} \bigg|_{\xi = l_r} = K u(t) \qquad (18)$$

Boundary condition of the isolated end

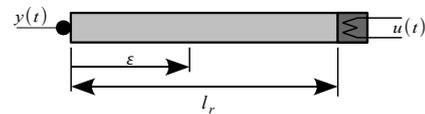$$\frac{\partial x(t, \xi)}{\xi} \bigg|_{\xi = 0} = 0 \qquad (19)$$

Figure 3. Model of metal rod

It can be shown that the plant is described by the following transfer function

$$P(s) = \frac{K \sqrt{\kappa}}{\sqrt{s} \sinh \left(l_r \sqrt{\dfrac{s}{\kappa}} \right)} \qquad (20)$$

where κ is th thermal conductivity constant, K is thermal transfer constant and l_r is length of the rod. These constant are chosen $\kappa = 0.0025$, $l = 0.5$, $K = 1$ in this example.

We consider design requirements to the maximum value of sensitivity $M_T = 1.4$ (i.e. $r \approx 1.46$, $c \approx -2.04$) and complementary sensitivity $M_S = 2$ (i.e. $r = 0.5$, $c = -1$) functions according to (10).

In the Fig. 5 the corresponding robustness regions are depicted. The red plot (the nearest one to the x-axis) corresponds to maximum value complementary sensitivity function requirement. The pair of controller parameter $k - k_i$ have to be chosen under the red plot to meet both design requirements.
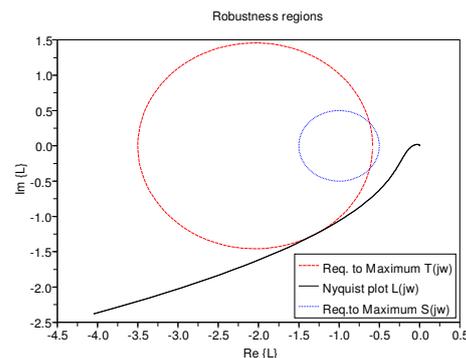

Figure 4. Open loop Nyquist plot
for metal rod model

In the Fig. 4 are shown design requirements and open-loop Nyquist plot for the chosen point $k = 2.722$, $k_i = 0.0193$ (marked with "x" in the Fig. 5).

Note that it could be specified others requirements for open loop transfer function. Every design requirement then determines some robustness region in the parameter plane. The intersection of all robustness regions satisfies all design requirements.
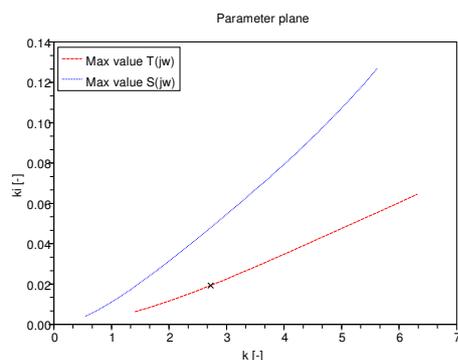
Figure 5. Parameter plane

CONCLUSION

The design method of robustness regions can be useful in practice and it can be used for wide spectrum of design problems for arbitrary process transfer function. Its main drawback is high demand for computation time. In the interactive design it is necessary to compute robustness regions repeatedly and it takes a lot of time on nowadays PC. This makes difficulties to use this method in practice. The method how simplify the computation and rapidly decrease the computation time was described in this paper.

ACKNOWLEDGMENTS

6 REFERENCES

Schlegel, M., Mertl J., Čech M. (2003). Generalized robustness regions for PID controllers. In: *Proces Control 2003*, p. 108-114, Slovac University of Technology, Bratislava

Neimark, Y. I. (1948). Structure of the D-partition of the space of polynomial and the diagram of Vishhnegradskii and Nyquist, *Dokl Akad Nauk SSSR, 59, 853*.

Shafiei, Z., Shenton, A. T. (1997). Frequency domain design of PID controllers for stable and unstable systems with time delay, In: *Automatica* 1997, Vol. 33, No 12, p. 2223-2232

Wankere R. M. (2001). Iterative Methods for Roots of Polynomials, [on-line]:
http://eprints.maths.ox.ac.uk/16/01/mekwi.pdf,
p. 13-14, University of Oxford

APPENDIX

Coefficients of polynomial equation (16):

$$c_0 = (-b^2 b_1^2 - 2 a a_1 b b_1 - a^2 a_1^2) \omega^6 r^2 + (b^2 b_1^2 + 2 a a_1 b b_1 + a^2 a_1^2) c^2 \omega^6$$

$$c_1 = (-2 a b b_1^2 - 2 a a_1^2 b) c^2 \omega^5 + (2 a b^2 b_1 - 2 a_1 b^3) c^2 \omega^4$$

$$c_2 = \left[(-2 b^2 b_1^2 - 4 a a_1 b b_1 - 2 a^2 a_1^2) \omega^4 + \left((2 b^3 + 2 a^2 b) b_1 + 2 a a_1 b^2 + 2 a^3 a_1 \right) \omega^3 \right] r^2 +$$
$$+ \left((b^2 + a^2) b_1^2 + 2 a a_1 b b_1 + a_1^2 b^2 + 3 a^2 a_1^2 \right) c^2 \omega^4 + \left((-2 b^3 - 4 a^2 b) b_1 - 2 a^3 a_1 \right) c^2 \omega^3 +$$
$$+ (b^4 + a^2 b^2) c^2 \omega^2$$

$$c_3 = (-2 a b b_1^2 - 2 a a_1^2 b) c^2 \omega^3 + \left((4 a b^2 + 2 a^3) b_1 + 2 a^2 a_1 b \right) c^2 \omega^2 + (-2 a b^3 - 2 a^3 b) c^2 \omega$$

$$c_4 = \left[(-b^2 b_1^2 - 2 a a_1 b b_1 - a^2 a_1^2) \omega^2 + \left((2 b^3 + 2 a^2 b) b_1 + 2 a a_1 b^2 + 2 a^3 a_1 \right) \omega - b^4 - 2 a^2 b^2 - a^4 \right] r^2 +$$
$$+ (a^2 b_1^2 + a^2 a_1^2) c^2 \omega^2 + (-2 a^2 b b_1 - 2 a^3 a_1) c^2 \omega + (a^2 b^2 + a^4) c^2$$