Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

17th International Conference on Process Control 2009

Hotel Baník, Štrbské Pleso, Slovakia, June 9 – 12, 2009

ISBN 978-80-227-3081-5

http://www.kirp.chtf.stuba.sk/pc09

Editors: M. Fikar and M. Kvasnica

Bakošová, M., Puna, D., Vasičkaninová, A., Karšaiová, M.: Robust PI and PID Stabilization of a Chemical Reactor, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 17th International Conference on Process Control '09*, Štrbské Pleso, Slovakia, 443–452, 2009.

Full paper online: http://www.kirp.chtf.stuba.sk/pc09/data/abstracts/087.html

ROBUST PI AND PID STABILIZATION OF A CHEMICAL REACTOR

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Abstract: Possibility to stabilize open-loop unstable processes using robust static output feedback controllers was studied. The non-iterative algorithm based on solving of linear matrix inequalities was used for design of robust PID like controllers. The design procedure guaranteed with sufficient conditions the closed-loop robust quadratic stability and the guaranteed cost of control. Possibility to use robust PI and PID controllers for stabilization of a continuous stirred tank reactor was verified by simulations. Considered reactor with one first order exothermic reaction had two uncertain parameters: reaction rate constant and the reaction enthalpy. Furthermore, the reactor had multiple steady states and it was stabilized in the surroundings of its open-loop unstable steady state. Simulation results confirmed that presented procedure can be successfully used for the design of robust stabilizing PID controllers.

Keywords: Robust stabilization, static output feedback, PID controller, continuous stirred tank reactor, multiple steady states.

1. INTRODUCTION

Continuous stirred tank reactors are ones of the most important plants in process industry and exothermic reactors are very interesting systems from the control viewpoint because of their potential safety problems and the possibility of exotic behavior such as multiple steady states, see e.g. Molnár et al. (2002). Furthermore, operation of chemical reactors is corrupted by many different uncertainties. Some of them arise from varying or not exactly known parameters, as e.g. reaction rate constants, reaction enthalpies, heat transfer coefficients, etc. Operating points of reactors change in other cases. All these facts can cause poor performance or even instability of closed-loop control systems. Optimal control strategies, which are often used for reactor control design, can fail in the presence of uncertainties. Application of robust control approach is one way for overcoming all these problems, as it is shown e.g. in Alvarez-Ramirez and Femat (1999), Gerhard et al. (2004), Bakošová et al. (2005), Tlacuahuac et al. (2005) and others.

Robust control has grown as one of the most important areas in modern control design since works by Doyle and Stein (1981), Zames and Francis (1983) and many others. One of the solved problems is also the problem of robust static output feedback control (RSOFC), which has been till now an important open question in control engineering, see e.g. Iwasaki et al. (1994), Syrmos et al. (1997) and references therein. Various approaches have been used to study two aspects of the robust stabilization problem.

The first aspect is related to conditions under which the linear system described in the state space can be stabilized via output feedback. The necessary and the sufficient conditions for stabilization of a linear continuous-time invariant system via static output feedback can be found e.g. in Kučera and de Souza (1995) and for stabilization of an uncertain affine linear systems e.g. in Veselý (2004). Recently, it has been shown that an extremely wide array of robust controller design problems can be reduced to the linear matrix inequality (LMI) problems. Especially, the LMIs in semi-definite programming attract a big interest because of their ability to describe non-trivial control design problems integrating various specifications such as robustness, structural and performance constraints, as well as their suitability for efficient numerical processing through various available solvers, see e.g. Boyd et al. (1994) and references therein.

The second aspect of the robust stabilization problem is related to a procedure for obtaining a stabilizing or robustly stabilizing control law. Most of recent works present iterative algorithms in which sets of LMI problems are repeated until certain convergence criteria are met, see e.g. Cao and Sun (1998), J. Bernussou and Korogui (2005).

The necessary and the sufficient conditions for stabilization of an uncertain polytopic system using static output feedback are formulated in this paper at first. The polytopic uncertainty is considered, while it is recognized as one of the most difficult structured uncertainties. Then the problem of robust controller design is transformed to the LMI problems. A computationally simple LMI based non-iterative algorithm is presented, which enables designing robust static output feedback PID like controllers. The design procedure assures with sufficient conditions the quadratic stability of the closed-loop system and the guaranteed cost of control. Designed robust controllers are used for stabilization of the open-loop unstable continuous-time stirred tank reactor (CSTR) with two uncertain parameters.

2. ROBUST STATIC OUTPUT FEEDBACK CONTROL

2.1 *Static output feedback, quadratic stability and guaranteed cost*

Consider a linear time-invariant (LTI) system given by the state-space representation

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x_0} \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) \end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the control, $\boldsymbol{y}(t) \in \mathbb{R}^r$ is the output and matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ have appropriate dimensions.

2.1.1. *Static output feedback with P controller* For the system (1), it is necessary to find a static output feedback

$$\boldsymbol{u}(t) = \boldsymbol{F}\boldsymbol{y}(t) \tag{2}$$

with $\boldsymbol{F} \in \mathbb{R}^{m \times r}$ such that the closed-loop system

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})\,\boldsymbol{x}(t) = \boldsymbol{A}_{CL}\boldsymbol{x}(t) \tag{3}$$

is stable, i.e. eigenvalues of $\boldsymbol{A}_{CL}$ have negative real parts.

Finding of $\boldsymbol{F}$ is important when the state matrix $\boldsymbol{A}$ is unstable since having $\boldsymbol{F}$ leads to a stabilizing static output feedback.

But, the output feedback (2) does not have an integral action. One way of forcing an integral action to the output feedback is to put a set of integrators at the output of the plant, see e.g. Mikleš et al. (2006), Puna and Bakošová (2007). Forcing of derivative action to the output feedback has analogous basement.

2.1.2. *Static output feedback with PI controller* For the system (1), it is necessary to find a static output feedback

$$\boldsymbol{u}(t) = \boldsymbol{F_1}\boldsymbol{y}(t) + \boldsymbol{F_2}\int_0^t \boldsymbol{y}(\tau)\mathrm{d}\tau \tag{4}$$

with $\boldsymbol{F}_1, \boldsymbol{F}_2 \in \mathbb{R}^{m \times r}$.

Let us define a new state $\boldsymbol{z}(t) = [\boldsymbol{z}_1^T(t), z_2^T(t)]^T$, where $\boldsymbol{z}_1(t) = \boldsymbol{x}(t)$ and $\boldsymbol{z}_2(t) = \int_0^t \boldsymbol{y}(\tau)\mathrm{d}\tau$. The dynamics of the newly defined system can by described as follows

$$\dot{\boldsymbol{z}}_1(t) = \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{z}_1(t) + \boldsymbol{B}\boldsymbol{u}(t),\ \boldsymbol{x}(t_0) = \boldsymbol{x_0} \tag{5}$$

$$\dot{\boldsymbol{z}}_2(t) = \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{z}_1(t) \tag{6}$$

or

$$\dot{\boldsymbol{z}}(t) = \overline{\boldsymbol{A}}\boldsymbol{z}(t) + \overline{\boldsymbol{B}}\boldsymbol{u}(t), \tag{7}$$

where

$$\overline{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \end{pmatrix}, \overline{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{pmatrix}. \tag{8}$$

The output of the newly defined system can by described as follows

$$\overline{\boldsymbol{y}}(t) = \overline{\boldsymbol{C}}\boldsymbol{z}(t), \tag{9}$$

where

$$\overline{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \tag{10}$$

with $\boldsymbol{I} \in \mathbb{R}^{r \times r}$.

So, the design of the static output feedback PI controller (4) is transformed to the design of a static output feedback P controller $\boldsymbol{F} = [\boldsymbol{F}_1 \quad \boldsymbol{F}_2]$ for the system (7) and (9).

2.1.3. *Static output feedback with PID controller* For the system (1), it is necessary to find a static output feedback

$$\boldsymbol{u}(t) = \boldsymbol{F}_1\boldsymbol{y}(t) + \boldsymbol{F}_2\int_0^t \boldsymbol{y}(\tau)\mathrm{d}\tau + \boldsymbol{F}_3\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} \quad (11)$$

with $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3 \in \mathbb{R}^{m\times r}$.

Let us define a new state $\boldsymbol{z}(t) = [\boldsymbol{z}_1^T(t), z_2^T(t)]^T$, where $\boldsymbol{z}_1(t) = \boldsymbol{x}(t)$ and $\boldsymbol{z}_2(t) = \int_0^t \boldsymbol{y}(\tau)d\tau$. Using $\boldsymbol{z}(t)$ and (1) leads to

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{z}_1(t) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \end{pmatrix} \boldsymbol{z}(t), \quad (12)$$

$$\int_0^t \boldsymbol{y}(t)\mathrm{d}t = \boldsymbol{z}_2(t) = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \boldsymbol{z}(t), \quad (13)$$

$$\frac{\mathrm{d}\boldsymbol{y}(t)}{\mathrm{d}t} = \boldsymbol{C}\dot{\boldsymbol{x}}(t) = \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{C}\boldsymbol{B}\boldsymbol{u}(t) = \begin{pmatrix} \boldsymbol{C}\boldsymbol{A} & \boldsymbol{0} \end{pmatrix} \boldsymbol{z}(t) + \boldsymbol{C}\boldsymbol{B}\boldsymbol{u}(t). \quad (14)$$

After the substitution (12), (13) and (14) into (11) and under the assumption that the matrix $\boldsymbol{F}_4 = (\boldsymbol{I} - \boldsymbol{F}_3\boldsymbol{C}\boldsymbol{B})^{-1}$ exists, we obtain

$$\boldsymbol{u}(t) = \overline{\boldsymbol{F}}_1\overline{\boldsymbol{y}}_1(t) + \overline{\boldsymbol{F}}_2\overline{\boldsymbol{y}}_2(t) + \overline{\boldsymbol{F}}_3\overline{\boldsymbol{y}}_3(t) \quad (15)$$

where $\overline{\boldsymbol{y}}_i(t) = \overline{\boldsymbol{C}}_i\boldsymbol{z}(t)$, $i = 1, 2, 3$, $\overline{\boldsymbol{C}}_1 = (\boldsymbol{C} \quad \boldsymbol{0})$, $\overline{\boldsymbol{C}}_2 = (\boldsymbol{0} \quad \boldsymbol{I})$, $\overline{\boldsymbol{C}}_3 = (\boldsymbol{C}\boldsymbol{A} \quad \boldsymbol{0})$, $\overline{\boldsymbol{F}}_1 = \boldsymbol{F}_4\boldsymbol{F}_1$, $\overline{\boldsymbol{F}}_2 = \boldsymbol{F}_4\boldsymbol{F}_2$ and $\overline{\boldsymbol{F}}_3 = \boldsymbol{F}_4\boldsymbol{F}_3$.

Defining

$$\overline{\boldsymbol{F}} = \begin{pmatrix} \overline{\boldsymbol{F}}_1 & \overline{\boldsymbol{F}}_2 & \overline{\boldsymbol{F}}_3 \end{pmatrix}, \quad (16)$$

$$\overline{\boldsymbol{y}}(t) = \begin{pmatrix} \overline{\boldsymbol{y}}_1^T(t) & \overline{\boldsymbol{y}}_2^T(t) & \overline{\boldsymbol{y}}_3^T(t) \end{pmatrix}^T, \quad (17)$$

$$\overline{\boldsymbol{C}} = \begin{pmatrix} \overline{\boldsymbol{C}}_1^T & \overline{\boldsymbol{C}}_2^T & \overline{\boldsymbol{C}}_3^T \end{pmatrix}^T, \quad (18)$$

we obtain a new dynamic system

$$\dot{\boldsymbol{z}}(t) = \overline{\boldsymbol{A}}\boldsymbol{z}(t) + \overline{\boldsymbol{B}}\boldsymbol{u}(t)$$
$$\overline{\boldsymbol{y}}(t) = \overline{\boldsymbol{C}}\boldsymbol{z}(t) \quad (19)$$

with

$$\boldsymbol{u}(t) = \overline{\boldsymbol{F}}\overline{\boldsymbol{y}}(t) \quad (20)$$

and

$$\overline{\boldsymbol{A}} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{0} \end{pmatrix}, \overline{\boldsymbol{B}} = \begin{pmatrix} \boldsymbol{B} \\ \boldsymbol{0} \end{pmatrix}, \overline{\boldsymbol{C}} = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{C}\boldsymbol{A} & \boldsymbol{0} \end{pmatrix} \quad (21)$$

So, the design of a static output feedback PID controller is transformed to the design of a static output feedback P controller for the system (19). After finding the P controller described by $\overline{\boldsymbol{F}} = \begin{pmatrix} \overline{\boldsymbol{F}}_1 & \overline{\boldsymbol{F}}_2 & \overline{\boldsymbol{F}}_3 \end{pmatrix}$, we obtain the PID controller (11) as follows

$$\boldsymbol{F}_3 = \overline{\boldsymbol{F}}_3(\boldsymbol{I} + \boldsymbol{C}\boldsymbol{B}\overline{\boldsymbol{F}}_3)^{-1} \quad (22)$$

$$\boldsymbol{F}_2 = (\boldsymbol{I} - \boldsymbol{F}_3\boldsymbol{C}\boldsymbol{B})\overline{\boldsymbol{F}}_2 \quad (23)$$

$$\boldsymbol{F}_1 = (\boldsymbol{I} - \boldsymbol{F}_3\boldsymbol{C}\boldsymbol{B})\overline{\boldsymbol{F}}_1 \quad (24)$$

2.1.4. *Quadratic stability* The sufficient condition for the asymptotic stability of the system (3) is feasibility, i.e. the existence of a quadratic Ljapunov function

$$V(\boldsymbol{x}) = \boldsymbol{x}(t)^T\boldsymbol{P}\boldsymbol{x}(t), \quad \boldsymbol{P} > 0 \quad (25)$$

such that

$$\frac{\mathrm{d}V(\boldsymbol{x}(t))}{\mathrm{d}t} = \boldsymbol{x}^T(t)[(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{F}\boldsymbol{C})]\boldsymbol{x}(t) < 0 \quad (26)$$

along all state trajectories. If a positive definite matrix $\boldsymbol{P}$ satisfying (26) exists, the system (3) is quadratically stable.

The necessary and the sufficient condition for quadratic stability of (3) is

$$\boldsymbol{A}_{CL}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{CL} < 0, \ \ \boldsymbol{P} > 0, \ \ \boldsymbol{P} = \boldsymbol{P}^T. \quad (27)$$

For $\boldsymbol{S} = \boldsymbol{P}^{-1}$, (27) can be rewritten as

$$\boldsymbol{S}\boldsymbol{A}_{CL}^T + \boldsymbol{A}_{CL}\boldsymbol{S} < 0, \ \ \boldsymbol{S} > 0, \ \ \boldsymbol{S} = \boldsymbol{S}^T. \quad (28)$$

The problem of solving the matrix inequality (28) is difficult because it is not jointly convex problem.

2.1.5. *Guaranteed cost* Suppose the cost function associated with the system (1) in the form

$$J = \int_0^\infty \left[\boldsymbol{x}(t)^T\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}(t)^T\boldsymbol{R}\boldsymbol{u}(t)\right] dt \quad (29)$$

where $\boldsymbol{Q} = \boldsymbol{Q}^T \geq 0$ and $\boldsymbol{R} = \boldsymbol{R}^T > 0$ are matrices of appropriate dimensions.

If there exist a control law $\boldsymbol{u}^*(t)$ and a positive scalar J^* such that the closed loop system (3) is stable, and the cost function (29) satisfies $J \leq J^*$, then J^* is said to be guaranteed cost, and $\boldsymbol{u}^*(t)$ is said to be guaranteed cost control law for the system (1), see e.g. Veselý (2002).

2.2 *Robust static output feedback, robust quadratic stability and guaranteed cost*

Consider again the linear time-invariant system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) \quad (30)$$

Suppose further that the system (30) is a polytop of linear time-invariant systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$
$$\boldsymbol{y}(t) = \boldsymbol{C}_i\boldsymbol{x}(t) \tag{31}$$
$$i = 1, \dots, N$$

which represent vertices of (30) and matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are convex envelopes of matrices $\boldsymbol{A}_i$, $\boldsymbol{B}_i$, $\boldsymbol{C}_i$, respectively, $i = 1, \dots, N$. The number of vertex systems $N = 2^p$, where p is the number of uncertain parameters of (30).

Consider also an uncertain polytopic closed-loop system

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} + \boldsymbol{BFC})\,\boldsymbol{x}(t) = \boldsymbol{A}_{CL}\boldsymbol{x}(t) \tag{32}$$

with a static output feedback controller $\boldsymbol{F}$, where $\boldsymbol{A}_{CL}$ is a convex envelope of a set of linear time invariant matrices $\boldsymbol{A}_{CLi}$

$$\boldsymbol{A}_{CLi} = \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}\boldsymbol{C}_i,\ i = 1, \dots, N. \tag{33}$$

System (32) is quadratically stable if and only if there exists a positive definite matrix $\boldsymbol{P} > 0$ such that following inequalities hold

$$\boldsymbol{A}_{CLi}^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_{CLi} < 0, \quad \boldsymbol{P} > 0,\ i = 1, \dots, N. \tag{34}$$

Consider the uncertain polytopic system (30). Then according to Veselý (2002), the following two statements are equivalent.

(1) The system (30) is robust static output feedback quadratically stabilizable.
(2) There exist a positive definite matrix $\boldsymbol{P} = \boldsymbol{P}^T > 0$ and a matrix $\boldsymbol{F}$ satisfying the following matrix inequalities

$$(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}\boldsymbol{C}_i)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}\boldsymbol{C}_i) < 0$$
$$i = 1, \dots, N. \tag{35}$$

Consider the uncertain polytopic system (30). Then according to Veselý (2002), the following three statements are equivalent.

(1) The system (30) is simultaneously static output feedback stabilizable with guaranteed cost

$$\int_0^\infty \left[\boldsymbol{x}(t)^T\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}(t)^T\boldsymbol{R}\boldsymbol{u}(t)\right] dt \le$$
$$\boldsymbol{x}_0(t)^T\boldsymbol{P}\boldsymbol{x}_0(t) = J^*, \quad \boldsymbol{P} > 0. \tag{36}$$

(2) There exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$ and a matrix $\boldsymbol{F}$ such that the following inequalities hold

$$(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}\boldsymbol{C}_i)^T\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F}\boldsymbol{C}_i) + \boldsymbol{Q}$$
$$+ \boldsymbol{C}_i^T\boldsymbol{F}^T\boldsymbol{R}\boldsymbol{F}\boldsymbol{C}_i < 0,\ i = 1, \dots, N. \tag{37}$$

(3) There exist matrices $\boldsymbol{P} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{R} > 0$ and a matrix $\boldsymbol{F}$ such that the following inequalities hold

$$\boldsymbol{A}_i^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i - \boldsymbol{P}\boldsymbol{B}_i\boldsymbol{R}^{-1}\boldsymbol{B}_i^T\boldsymbol{P}$$
$$+ \boldsymbol{Q} \le 0, \quad i = 1, \dots, N \tag{38}$$
$$(\boldsymbol{B}_i^T\boldsymbol{P} + \boldsymbol{R}\boldsymbol{F}\boldsymbol{C}_i)\boldsymbol{\phi}_i^{-1}(\boldsymbol{B}_i^T\boldsymbol{P} + \boldsymbol{R}\boldsymbol{F}\boldsymbol{C}_i)^T$$
$$- \boldsymbol{R} \le 0, \quad i = 1, \dots, N \tag{39}$$

where

$$\boldsymbol{\phi}_i = -(\boldsymbol{A}_i^T\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_i - \boldsymbol{P}\boldsymbol{B}_i\boldsymbol{R}^{-1}\boldsymbol{B}_i^T\boldsymbol{P}$$
$$+ \boldsymbol{Q}), \quad i = 1, \dots, N. \tag{40}$$

2.3 *Robust static output feedback controller design*

The design procedure for simultaneous static output feedback stabilization of the system (30) with guaranteed cost (36) is according to Veselý (2002) based on statements formulated in previous sections.

Using the Schur complement formula and defining $\boldsymbol{S} = \boldsymbol{P}^{-1}$, the inequalities (38) are transformed to the following LMIs

$$\begin{bmatrix} \boldsymbol{S}\boldsymbol{A}_i^T + \boldsymbol{A}_i\boldsymbol{S} - \boldsymbol{B}_i\boldsymbol{R}^{-1}\boldsymbol{B}_i^T & \boldsymbol{S}\sqrt{\boldsymbol{Q}} \\ \sqrt{\boldsymbol{Q}}\boldsymbol{S} & -\boldsymbol{I} \end{bmatrix} \le 0$$
$$\gamma\boldsymbol{I} < \boldsymbol{S}, \quad i = 1, \dots, N \tag{41}$$

where $\gamma > 0$ is any non-negative constant.

Using $\boldsymbol{P} = \boldsymbol{S}^{-1}$, the inequalities (39) can be rewritten to the following LMIs

$$\begin{bmatrix} -\boldsymbol{R} & \boldsymbol{B}_i^T\boldsymbol{P} + \boldsymbol{R}\boldsymbol{F}\boldsymbol{C}_i \\ (\boldsymbol{B}_i^T\boldsymbol{P} + \boldsymbol{R}\boldsymbol{F}\boldsymbol{C}_i)^T & -\boldsymbol{\phi}_i \end{bmatrix} \le 0$$
$$i = 1, \dots, N. \tag{42}$$

The algorithm for static output simultaneous stabilization of the system (30) with the guaranteed cost (36) is following.

(1) Compute $\boldsymbol{S} = \boldsymbol{S}^T > 0$ from the LMIs (41).
(2) $\boldsymbol{P} = \boldsymbol{S}^{-1}$.
(3) Compute $\boldsymbol{F}$ from the LMIs (42).
(4) If the solution of (41) is not feasible, the system (30) is not simultaneously stabilizable by a static output feedback. If the solution of (42) is not feasible, the closed-loop system (32) is not quadratically stable with guaranteed cost. Then change $\boldsymbol{Q}$, $\boldsymbol{R}$ or γ in order to find feasible solutions. If the solutions of (41), (42) are feasible, then the system (30) is simultaneously stabilizable and the system (32) is quadratically stable with guaranteed cost $J^* = \boldsymbol{x}_0^T\boldsymbol{P}\boldsymbol{x}_0$.

There are two parameters in the presented algorithm, which can be called tuning parameters. They are weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ in (36). The choice of γ in (41) also influences the solution, but γ is only a LMI variable.

3. CONTROLLED CSTR

Hydrolysis of propylene oxide to propylene glycol in a continuous stirred tank reactor (CSTR) (Molnár et al. (2002)) was chosen as a controlled process. The reaction is as follows

$$C_3H_6O + H_2O \longrightarrow C_3H_8O_2 \tag{43}$$

and it is of the first order with respect to propylene oxide as a key component. The dependence of the reaction rate constant on the temperature in the CSTR is described by the Arrhenius equation

$$k = k_\infty e^{-\frac{E_a}{RT_r}} \tag{44}$$

where k_∞ is the pre-exponential factor, E_a is the activation energy, R is the universal gas constant and T_r is the temperature of the reaction mixture. Assuming usual simplifications (Ingham et al. (1994)), the mass balance for any species j in the reactor is

$$V_r \frac{dc_j}{dt} = q_r (c_{0j} - c_j) + \nu_j r V_r \tag{45}$$

The simplified enthalpy balance of the reaction mixture is

$$V_r \rho_r C_{Pr} \frac{dT_r}{dt} = q_r \rho_r C_{Pr} (T_{r0} - T_r) - UA (T_r - T_c) + r V_r (-\Delta_r H^o) \tag{46}$$

and the simplified enthalpy balance of the cooling medium is

$$V_c \rho_c C_{Pc} \frac{dT_c}{dt} = q_c \rho_c C_{Pc} (T_{c0} - T_c) + UA (T_r - T_c) \tag{47}$$

In previous balances, V is the volume, c is the molar concentration, q is the volumetric flow rate, ν is the stoichiometric coefficient, $r = kc_{C_3H_6O}$ is the molar rate of the chemical reaction, T is the temperature, ρ is the density, C_P is the specific heat capacity, $\Delta_r H^o$ is the reaction enthalpy, U is the overall heat transfer coefficient and A is the heat exchange area. The subscripts denote: 0 the feed, r the reaction mixture, c the cooling medium, and j the j–th component. The values of constant parameters and steady-state inputs of the CSTR are summarized in Table 1.

Model uncertainties of the CSTR follow from the fact that there are two physical parameters in this reactor, which values are known within intervals: the reaction enthalpy and the pre-exponential factor (Table 2). The nominal values of these parameters are mean values of the intervals. The minimum and the maximum values of the intervals are used for obtaining models, which create the vertex systems (31) of the uncertain polytopic system (30).

Table 1. Constant parameters and steady-state inputs of the CSTR

Variable	Value	Unit
V_r	2.407	m^3
V_c	2	m^3
ρ_r	947.19	kg m^{-3}
ρ_c	998	kg m^{-3}
C_{Pr}	3.7187	kJ $kg^{-1}K^{-1}$
C_{Pc}	4.182	kJ $kg^{-1}K^{-1}$
AU	120	kJ $min^{-1}K^{-1}$
E_a/R	10183	K
q_r	0.072	m^3min^{-1}
q_c	0.6307	m^3min^{-1}
$c_{C_3H_6O,0}$	0.0824	kmol m^{-3}
$c_{C_3H_8O_2,0}$	0	kmol m^{-3}
T_{r0}	299.05	K
T_{c0}	288.15	K

Table 2. Uncertain parameters in the CSTR

Parameter	Unit	Minimal Value	Maximal Value
$\Delta_r H^o$	kJ $kmol^{-1}$	-5.28×10^6	-5.64×10^6
k_∞	min^{-1}	2.4067×10^{11}	3.2467×10^{11}

4. SIMULATION RESULTS

4.1 *Steady-state and open-loop analysis*

The steady-state behavior of the chemical reactor with nominal values and also with all 4 combinations of minimal and maximal values of 2 uncertain parameters was studied at first. It can be stated the reactor has always three steady states, two of them are stable and one is unstable. The situation for the nominal model is shown in Figure 1, where the curve Q_{GEN} (red line) is the heat generated by the reaction and the line Q_{OUT} (blue line) is the heat withdrawn from the reactor. The steady-state operating points of the reactor are points, where the curve and the line intersect. The steady states are stable if the slope of the cooling line is higher than the slope of the heat generated curve. This condition is satisfied in steady states at the temperatures $T_r = 296.7$ K and $T_r = 377.5$ K, and it is not satisfied in the steady state at the $T_r = 343.1$ K. The steady-state behavior of the chemical reactor is similar for all vertex systems.

From the viewpoint of safety operation or in the case when the unstable steady state coincides with the point that yields the maximum reaction rate at a prescribed temperature, it is necessary to stabilize CSTR in the surroundings of the open-loop unstable steady state, see e. g. Pedersen and Jorgensen (1999), Antonelli and Astolfi (2003), González and Alvarez (2006), Salgado et al. (2006), Salau et al. (2006).

In this context, the open-loop behavior of the reactor was studied at first. The initial temperature of the reaction mixture was chosen $T_r(0) = 341.5$ K. Simulation results obtained for the nominal model (black line) and 4 vertex systems (ma-

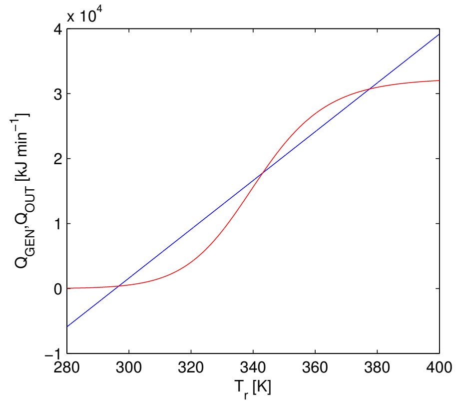

Fig. 1. Multiple steady states of CSTR

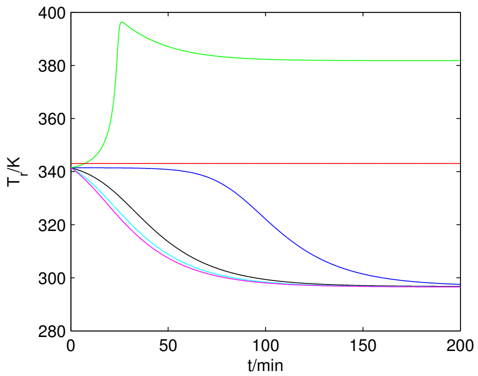

Fig. 2. Open-loop response of CSTR

genta, blue, cyan and green lines) are shown in Figure 2. They confirm that the temperature of the reaction mixture in the CSTR does not converge without feedback control into the unstable steady state represented by the temperature $T_r = 343.1$ K (red line).

4.2 *Stabilization of the CSTR*

The main aim was to stabilize the CSTR using robust static output feedback into its open-loop unstable steady state. The design of robust stabilizing PI and PID controllers was based on the theory presented in Section 2.

It was necessary to obtain a linear state space model (30) of the controlled process at first. The linear mathematical model of the CSTR was derived using linearization of non-linear terms in the mass balances of propylene oxide and propylene glycol and the enthalpy balance of the reaction mixture. It was supposed for control purposes that the reactor was a two-input single-output system. The reaction mixture flow rate q_r and the coolant flow rate q_c were chosen as the control inputs and the temperature of the reaction mixture T_r was selected as the controlled output. The other input variables were constant. The matrices of the nominal linear model in the operating point at the temperature $T_r = 343.1$ K were

$$A_0 = \begin{pmatrix} -0.0664 & 0 & -0.0001 & 0 \\ 0.0365 & -0.0299 & 0.0001 & 0 \\ 54.9420 & 0 & 0.1329 & 0.0138 \\ 0 & 0 & 0.0144 & -0.3297 \end{pmatrix}$$
$$B_0 = \begin{pmatrix} 0.01886 & 0 \\ -0.0188 & 0 \\ -18.3005 & 0 \\ 0 & -1.1978 \end{pmatrix}$$
$$C_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \qquad (48)$$

The eigenvalues of $\boldsymbol{A}_0$ are -0.0299, 0.0929, -0.0260, -0.3301, and they confirm the instability of the reactor at the temperature $T_r = 343.1$ K. For 2 uncertain parameters, we obtained $2^2 = 4$ linear models, which represented vertices (31) of the uncertain polytopic system (30). All vertex systems were also unstable.

For finding stabilizing output feedback PI or PID controllers, it was necessary to solve two sets of LMIs (41), (42). For their solution, the LMI MATLAB Toolbox was used. Following parameters influenced solution and could be changed: $\boldsymbol{Q}, \boldsymbol{R}, \gamma$. In dependence on the choice of these parameters, it was possible to find several stabilizing PI and PID controllers. The best simulation results with fast responses and small overshoots were obtained using the PI and the PID controllers presented in Table 3. They were obtained for $\boldsymbol{Q}$, $\boldsymbol{R}$ and γ chosen as follows

$$\boldsymbol{Q} = \begin{pmatrix} 27 & 0 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 & 0 \\ 0 & 0 & 9\times10^{-7} & 0 & 0 \\ 0 & 0 & 0 & 9\times10^{-7} & 0 \\ 0 & 0 & 0 & 0 & 9\times10^{-7} \end{pmatrix}$$
$$R = \begin{pmatrix} 9\times10^{-3} & 0 \\ 0 & 9\times10^{-4} \end{pmatrix}, \gamma = 5\times10^{-6}.$$

The choice of $\boldsymbol{Q}$ and $\boldsymbol{R}$ was done according to the values of the state and the control variables. Because the values of these variables differ by several orders, the values of elements of $\boldsymbol{Q}$ and $\boldsymbol{R}$ also differ by several orders.

Table 3. Stabilizing PI and PID controllers

PI	$\begin{bmatrix} 3.0823\times10^{-2} & 8.5148\times10^{-3} \\ 5.4289\times10^{-1} & 2.2448\times10^{-1} \end{bmatrix}$
PID	$\begin{bmatrix} 9.7077\times10^{-2} & 3.5293\times10^{-2} & 1.4297\times10^{-2} \\ 4.8860 & 1.8347 & 3.6488 \end{bmatrix}$

The possibility to stabilize the reactor using designed robust static output feedback PI and PID controllers was studied by simulations. The nonlinear model of the CSTR was used as the controlled system and the initial temperature of the reaction mixture was $T_r(0) = 341.5$ K. The aim was to control the temperature in the CSTR to the value $T_r = 343.1$ K. The control input boundaries were as follows: $q_r \in [0; 0.18]$ $m^3 min^{-1}$ and $q_c \in [0; 1.58]$ $m^3 min^{-1}$. Simulation results obtained with the PI and PID robust static feedback controllers presented in Table 3 are shown in Figs. 3, 4 for the controlled output T_r and the control inputs q_r and q_c. The setpoint is drawn by red line, the black line represents the nominal system and vertex systems are represented by magenta, blue, cyan and green lines. Both, PI and PID static output feedback controllers are able to stabilize the CSTR with uncertainties into its open-loop unstable steady state.

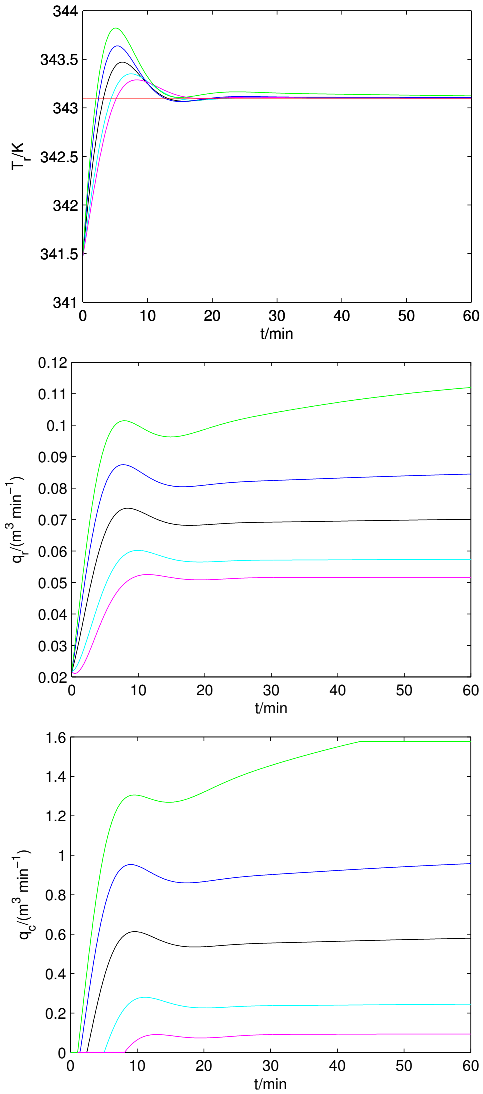

Fig. 3. Robust PI stabilization of CSTR

The possibility to use robust controllers in the presence of disturbances was studied, too. Following disturbances were loaded: the inlet temperature of the coolant T_{c0} decreased by 5 K for $t \in [50; 100)$ min, the feed temperature of the reaction mixture T_{r0} decreased by 3 K for $t \in [100; 150)$ min and the feed concentration of propylene oxide $c_{C_3H_6O,0}$ decreased by 0.006 mol m^{-3} for $t \in [150; 200)$ min. Obtained

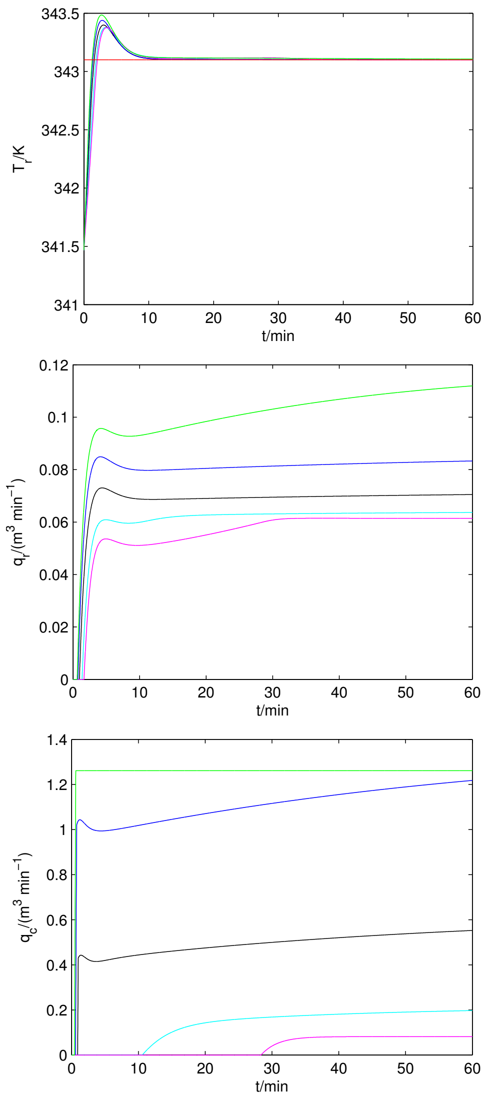

Fig. 4. Robust PID stabilization of CSTR

simulation results are shown in Figures 5, 6. The robust PI and PID controllers are able to stabilize the CSTR also in the presence of disturbances. The robust PID controller attenuates disturbances very fast and the overshoots caused by disturbances are very small.

The ability of robust static output feedback PI a PID controller to stabilize the CSTR with noisy signals was also studied. The white noise signal was generated using the Simulink block Band-limited White Noise and the noise power was 0.0005. The signal was added to the controlled output. Obtained simulation results are shown in Figures 7, 8. The robust PI and PID controllers are able to stabilize the CSTR with noisy signals, but especially PID controllers can generate the control inputs which cannot be realized.

CONCLUSION

Possibility to stabilize the exothermic CSTR with two uncertain parameters using static output

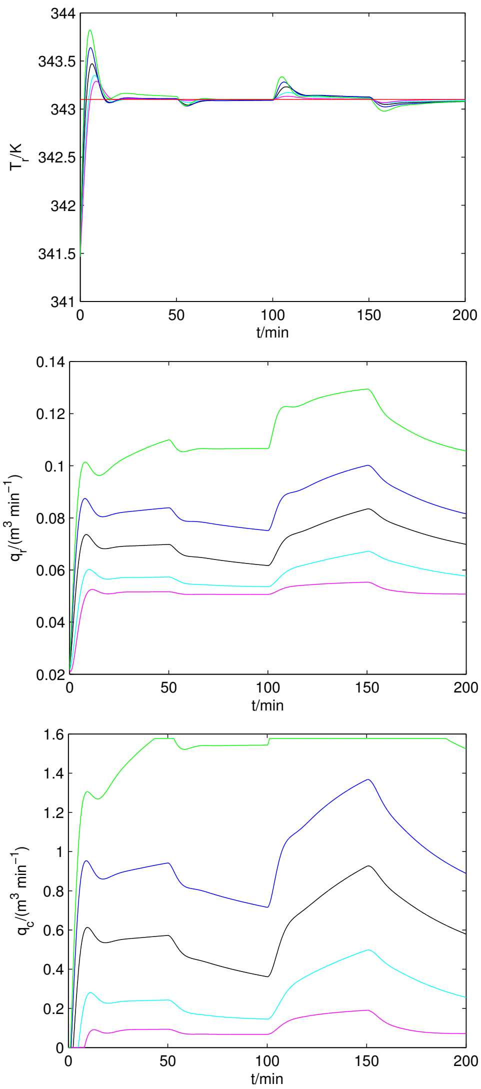

Fig. 5. Robust PI stabilization of CSTR in the presence of disturbances

feedback PI and PID controllers was studied. The results confirm that the presented simple non-iterative algorithm based on solving of two sets of LMIs is an effective tool for the design of robust stabilizing controllers. Its advantage is that it can be used for P, PI and PID controller design. Robust static output feedback PI or PID controllers can be successfully used for control of CSTRs with multiple steady states, uncertainties and disturbances, even though CSTRs are very complicated systems from the control viewpoint. Both, PI and PID controllers are able to stabilize the open-loop unstable processes and their advantage in comparison with the robust P controller is that they do not retain offsets. The disadvantage of PID controllers is their more complicated implementation and they are not suitable for using in the presence of noise.

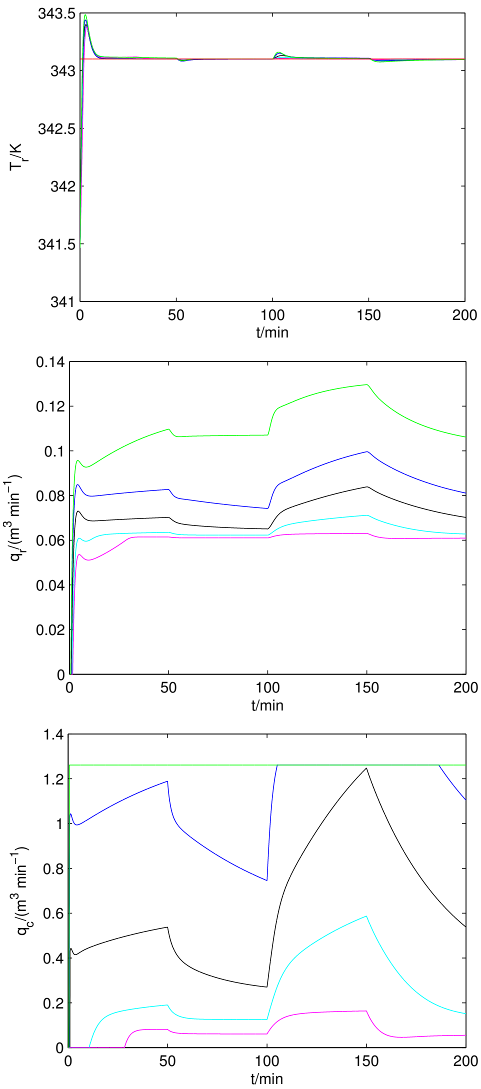

Fig. 6. Robust PID stabilization of CSTR in the presence of disturbances

ACKNOWLEDGMENTS

The work has been supported by the Scientific Grant Agency of the Slovak Republic under grants 1/4055/07, 1/0071/09 and by the Slovak Research and Development Agency under the project APVV-0029-07. This support is very gratefully acknowledged.

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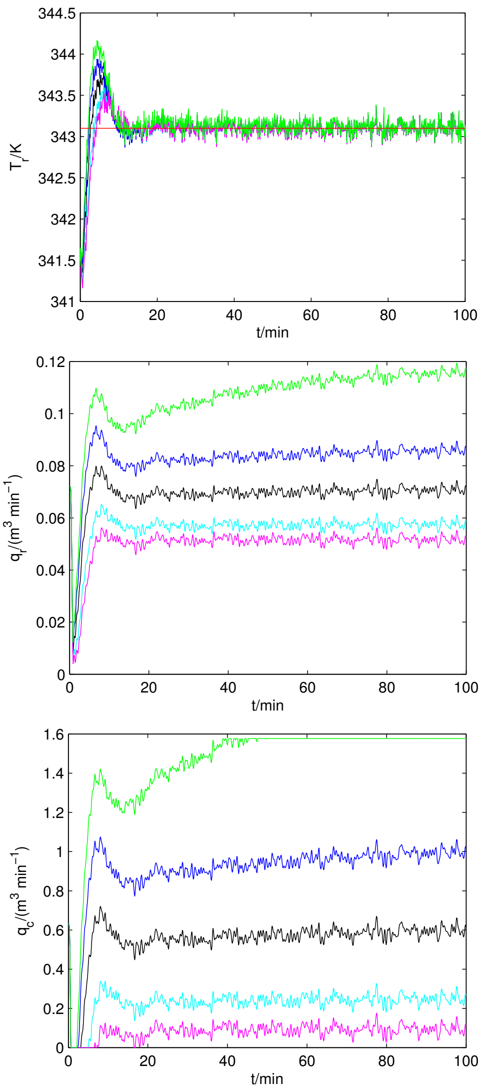

Fig. 7. Robust PI stabilization of CSTR with noisy signals

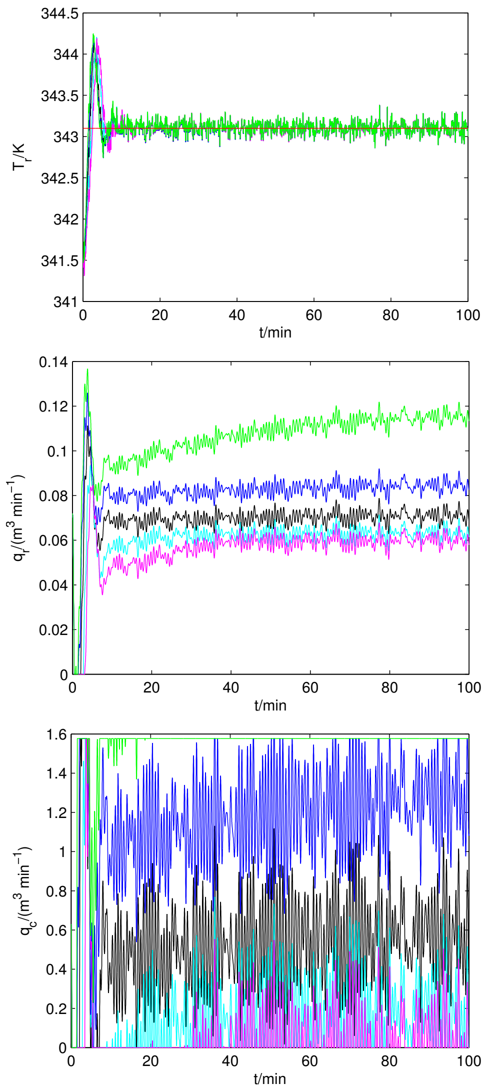

Fig. 8. Robust PID stabilization of CSTR with noisy signals

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