Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

of the 18$^{\text{th}}$ International Conference on Process Control

Hotel Titris, Tatranská Lomnica, Slovakia, June 14 – 17, 2011

ISBN 978-80-227-3517-9

http://www.kirp.chtf.stuba.sk/pc11

Editors: M. Fikar and M. Kvasnica

Kotta, Ü., Mullari, T., Shumsky, A., Zhirabok, A.: Accessibility and Feedback Linearization for SISO Discrete-Time Nonlinear Systems: New Tools, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 18th International Conference on Process Control*, Tatranská Lomnica, Slovakia, 153–161, 2011.

Full paper online: http://www.kirp.chtf.stuba.sk/pc11/data/abstracts/021.html

Accessibility and Feedback Linearization for SISO Discrete-Time Nonlinear Systems: New Tools ⋆

Ülle Kotta * **Tanel Mullari** * **Alexey Ye. Shumsky** ** **Alexey N. Zhirabok** ***

* *Institute of Cybernetics at Tallinn University of Technology, Akadeemia tee 21, 12618 Tallinn, Estonia (e-mail: kotta@cc.ioc.ee).*
** *Institute of Applied Mathematics, Far Eastern Branch of Academy of Sciences, Radio street 7, 690041 Vladivostok, Russia (e-mail: shumsky@mail.primorye.ru)*
*** *Dept. of Design and Technology of Radio Equipment, Far Eastern Federal University, Pushkinskaya street 10, 690950 Vladivostok, Russia (e-mail: zhirabok@mail.ru)*

Abstract: The tools of the algebra of functions are applied to readdress the accessibility and static state feedback linearization problems for discrete-time nonlinear control systems. These tools are also applicable for nonsmooth systems. Moreover, the close connections are established between the new results and those based on differential one-forms.

Keywords: Feedback linearization, state feedback, nonlinear control systems, discrete-time systems, algebraic approaches

1. INTRODUCTION

The approach based on the vector spaces of differential one-forms over suitable differential/difference fields of nonlinear functions offers the complementary (dual) tools to the differential geometric methods for studying the nonlinear control systems, either continuous- or discrete-time, see Conte et al. [2007]. These tools are characterized by their inherent simplicity, universality and strong similarity to their linear counterparts.

However, there exists another mathematical approach that relies on a certain algebraic structure, called the algebra of functions, see Zhirabok and Shumsky [2008]. The main idea for developing the algebra of functions traces back to the book by Hartmanis and Stearns [1966], who introduced the algebra of partitions for finite automata defined via the transition tables or graphs. In the algebra of functions the partitions were replaced by functions generating them and the analogous operations and operators for functions were introduced. The four key elements of the algebra of functions are partial preorder relation, binary operations (sum and product, defined in a specific manner), binary relation and certain operators **m** and **M**. The first two elements are defined on the arbitrary set of vector functions whereas the other two are defined for functions with the domain being the state space of the control system. Like the tools based on the differential forms, the algebra of functions provides a unified viewpoint to study the discrete-time as well as the continuous-time control systems; additionally it allows to address also the discrete-event systems like those in Shumsky and Zhirabok [2010a,b]. An important point to stress is that these tools (unlike most previous methods) do not require the system to be described in terms of smooth functions.

The goal of this paper is to compare the tools of the algebra of functions with those based on the differential forms. Our purpose is to compare the assumptions made on the control system, the basic algorithms and the solutions of few chosen control problems, like accessibility and static state feedback linearization. In order to focus on the key aspects and keep the presentation simple, we restrict ourselves in this paper to the discrete-time single-input systems.

Whereas the number of publications on the topic of static state feedback linearization is huge, the situation is different for the discrete-time case, see Jayaraman and Chizek [1993], Nam [1989], Grizzle [1986], Nijmeijer and van der Schaft [1990], Aranda-Bricaire et al. [1996], Jakubczyk [1987], Simões and Nijmeijer [1996]. Except Simões and Nijmeijer [1996], all papers focus on smooth feedback.

The interest in recasting these old problems is that the new solution is not based on the 'tangent linearized system' description of the system but is found directly by manipulating the functions on the system equation level. Therefore, for finding the solution one is not required to solve a partial differential equation or to integrate the differential one-forms. The new approach is based on the algebra of functions. Then we compare the new results with the one described in terms of the differential forms in Aranda-Bricaire et al. [1996].

⋆ This work was supported by the Estonian governmental funding project no. SF0140018s08, ESF grant no. 8365 and Russian Foundation of Basic Researchers Grants 10-08-00133 and 10-08-91220-CT.

2. TOOLS BASED ON DIFFERENTIAL FORMS

Consider a discrete-time nonlinear control system Σ of the form

$$\sigma(x) = f(x,u), \tag{1}$$

where by $\sigma(x)$ is denoted the forward shift of x, alternatively written as x^+, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, the variables $x = [x_1, ..., x_n]^T$ [1] and u are the coordinates of the state space $\mathbb{R}^n$ and the input space $\mathbb{R}$, respectively. In the approach based of differential one-forms, one assumes that f in (1) is analytic function. However, in the approach based on the algebra of functions there is no need to assume that f is analytic. Actually, f is allowed even to be nonsmooth.

In the study of discrete-time nonlinear control systems the following assumption is usually made, that guarantees the forward shift operator, defined by equation (1), to be injective. Note that this assumption is not restrictive since it is always satisfied for accessible systems, see Grizzle [1993].

Assumption 1. The system (1) is submersive, i. e. generically, $\text{rank}[\partial f(x,u)/\partial(x,u)] = n$.

Note that this assumption is not restrictive, especially for problems studied in this paper, since by the results of Grizzle [1993], submersivity is a necessary condition for a system to be accessible. Moreover, accessibility is a necessary condition for static state feedback linearizability.

In the approach of differential one-forms one associates with system (1) an inversive difference field $(\mathcal{K}, \sigma)$ of meromorphic functions in a finite number of independent system variables, see Aranda-Bricaire et al. [1996]. The forward shift operator $\sigma : \mathcal{K} \to \mathcal{K}$ is defined by

$$\sigma\varphi(x,u) = \varphi(\sigma(x), \sigma(u)) = \varphi(f(x,u), \sigma(u)).$$

However, not every element in $\mathcal{K}$ has necessarily a preimage with respect to σ. To guarantee that σ is an automorphism, one has to extend equations (1) by

$$\tilde{x} = g(x,u) \tag{2}$$

such that

$$\text{rank}\frac{\partial(f^T, g^T)^T}{\partial(x,u)} = n + 1.$$

Though the choice of the function $g(x,u)$ is not unique, all choices lead to isomorphic differential fields. In what follows we use sometimes the abridged notation $\varphi^+ = \sigma(\varphi)$ and $\varphi^- = \sigma^{-1}(\varphi)$ for $\varphi \in \mathcal{K}$.

Over the field $\mathcal{K}$ one can define a vector space $\mathcal{E} := \text{span}_{\mathcal{K}}\{\text{d}\varphi \mid \varphi \in \mathcal{C}\}$ spanned by the differentials of the elements of $\mathcal{C} = \{x, \sigma^k(u), k \geq 0\}$. The elements of $\mathcal{E}$ are called differential one-forms. The forward shift operator $\sigma : \mathcal{K} \to \mathcal{K}$ induces a forward shift operator $\sigma : \mathcal{E} \to \mathcal{E}$ by

$$\Sigma_i a_i \text{d}\varphi_i \to \Sigma_i a_i^+ \text{d}(\sigma(\varphi_i)).$$

A 1-form $\omega \in \mathcal{E}$ is called exact if $\text{d}\omega = 0$ and closed if $\text{d}\omega \wedge \omega = 0$, where $\wedge$ denotes the wedge product. The subspace of 1-forms in $\mathcal{E}$ is called completely integrable if it admits a basis which consists only of closed one-forms. The relative degree r of a 1-form ω in $\mathcal{X} := \text{span}_{\mathcal{K}}\{\text{d}x\}$ is defined by $r = \min\{k \in \mathbb{N} \mid \omega(k) \notin \mathcal{X}\}$.

Define a sequence of codistributions $\mathcal{H}_k$ as follows

$$\begin{aligned} \mathcal{H}_1 &= \text{span}_{\mathcal{K}}\{\text{d}x\} \\ \mathcal{H}_{k+1} &= \{\omega \in \mathcal{H}_k \mid \omega^+ \in \mathcal{H}_k\}, \quad k \geq 1. \end{aligned}$$

Each $\mathcal{H}_k$ contains the one-forms with relative degree equal to k or greater than k. The sequence $\mathcal{H}_k$ is non-increasing. There exists an integer $k^* \leq n$ such that for $1 \leq k \leq k^*$, $\mathcal{H}_{k+1} \subset \mathcal{H}_k$, $\mathcal{H}_{k^*+1} \neq \mathcal{H}_{k^*}$ but $\mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2} := \mathcal{H}_\infty$. Obviously, k^* is the minimal integer satisfying $\mathcal{H}_{k^*+1} = \mathcal{H}_{k^*+2}$ and $\mathcal{H}_\infty$ is the maximal codistribution, invariant with respect to the forward shift. Finally, note that the subspaces are invariant with respect to the regular static state feedback and state coordinate transformation Aranda-Bricaire et al. [1996].

3. THE ALGEBRA OF FUNCTIONS

To readdress accessibility and feedback linearization problems, the mathematical technique called the algebra of functions and developed in Zhirabok and Shumsky [2008] will be used. We recall below briefly the definitions and concepts to be used in this paper, see also Shumsky [2009]. Since these tools are not widely known, we provide many illustrative examples to illustrate the definitions.

The elements of algebra of functions are vector functions and its main ingredients are:

(1) relation of partial preorder, denoted by $\leq$,
(2) binary operations, denoted by $\times$ and $\oplus$,
(3) binary relation, denoted by Δ,
(4) operators **m** and **M**.

The first two elements are defined on the arbitrary set S of vector functions whereas the last two are defined for the set S_X of vector functions with the domain being the state space X.

Definition 1. (Relation of partial preorder) Given $\alpha, \beta \in S$, one says that $\alpha \leq \beta$ iff there exists a function γ such that

$$\beta(s) = \gamma(\alpha(s))$$

for $\forall s \in S$.

The definition means that every component of the function β can be expressed as a function of α. Clearly, $\alpha \leq \beta$ iff

$$\text{rank}[\partial\alpha/\partial s] = \text{rank}\begin{bmatrix}\partial\alpha/\partial s \\ \partial\beta/\partial s\end{bmatrix}.$$

Example 2. Let $\alpha(s) = [s_1, s_2]^T$, $\beta(s) = [s_1, s_1 s_2]^T$. Then $\alpha \leq \beta$ since there exists $\gamma(\alpha) = [\alpha_1, \alpha_1\alpha_2]^T$ such that $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1\alpha_2$. The inequality $\beta \leq \alpha$ does not hold in general, since $\alpha_2 = \beta_2/\beta_1$ is not valid for $s_1 = \beta_1(s) = 0$, i.e. on the set of measure zero.

If $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, then α and β are said to be incomparable.

Example 3. Let $\alpha(s) = [s_1 s_3, s_2]^T$ and $\beta(s) = [s_1, s_2 s_3]^T$; $\alpha(s)$ and $\beta(s)$ are incomparable. Note that $\alpha(s) \not\leq \beta(s)$, since

[1] Note that we often omit the symbol of transposition T in [,]^T for simplicity of presentation.

$$\text{rank}\left(\frac{\partial\alpha}{\partial s}\right) = \begin{pmatrix} s_3 & 0 & s_1 \\ 0 & 1 & 0 \end{pmatrix} \neq \text{rank}\begin{pmatrix} \partial\alpha/\partial s \\ \partial\beta/\partial s \end{pmatrix}$$
$$= \begin{pmatrix} s_3 & 0 & s_1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & s_3 & s_2 \end{pmatrix}.$$

In the similar manner one can show that $\beta(s) \leq \alpha(s)$.

Definition 4. (Strict equivalence) If $\alpha \leq \beta$ and $\beta \leq \alpha$, then α and β are called strictly equivalent, denoted by $\alpha \cong \beta$.

Note that the relation $\cong$ is reflexive, symmetric and transitive. The equivalence relation divides the set S into the *equivalence classes* containing the equivalent functions.

Example 5. The functions $\alpha(s) = [s_1, s_2]^T$ and $\beta(s) = [s_1, s_1 + s_2]^T$ are strictly equivalent since $\beta_1 = \alpha_1$, $\beta_2 = \alpha_1 + \alpha_2$, and $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2 - \beta_1$.

Besides the strict equivalence, we use the notion of equivalence, corresponding to the situation when one of the inequalities $\alpha \leq \beta$ or $\beta \leq \alpha$ may be violated on a set of measure zero.

Example 6. (Continuation of Example 2) The functions α and β are equivalent though not strictly equivalent.

Definition 7. Given $\alpha, \beta \in S$,
$$\alpha \times \beta = \max(\gamma \in S \mid \gamma \leq \alpha, \gamma \leq \beta),$$
and
$$\alpha \oplus \beta = \min(\gamma \in S \mid \alpha \leq \gamma, \beta \leq \gamma).$$

It follows from these definitions that the function $\alpha \times \beta$ is a maximal bottom of the functions α and β while $\alpha \oplus \beta$ is their minimal top. In the simple cases the definition may be used to compute $\alpha \oplus \beta$. For the general case, see Zhirabok and Shumsky [2008].

The rule for operation $\times$ is simple
$$(\alpha \times \beta)(s) = \begin{bmatrix} \alpha(s) \\ \beta(s) \end{bmatrix}.$$

However, the product may contain redundant (functionally dependent) components that have to be found and removed. Moreover, to simplify the computations, one is advised to replace the remaining components by equivalent but more simple functions. At moment, no algorithm exists for these two steps.

Example 8. (Computation of the functions $\alpha \times \beta$ and $\alpha \oplus \beta$). Let $S = \mathbb{R}^3$,
$$\alpha(s) = \begin{bmatrix} s_1 + s_2 \\ s_3 \end{bmatrix}, \quad \beta(s) = \begin{bmatrix} s_1 s_3 \\ s_2 s_3 \end{bmatrix}.$$

Then $(\alpha \times \beta)(s) \cong [s_1 + s_2, s_3, s_1 s_3]^T$, $(\alpha \oplus \beta)(s) \cong s_3(s_1 + s_2)$.

Definition 9. (Binary relation Δ) Given $\alpha, \beta \in S_X$, α and β are said to form an (ordered) pair, denoted as $(\alpha, \beta) \in \Delta$ if there exists a function f_* such that
$$\beta(f(x,u)) = f_*(\alpha(x), u) \quad (3)$$
for all $(x, u) \in X \times U$.

The example below shows that the binary relation is not symmetric.

Example 10. Let $\alpha(x) = x_2$, $\beta(x) = x_1$, and the state transition map in (1)
$$f(x,u) = \begin{bmatrix} \varphi_1(x_2, u) \\ \varphi_2(x_1, x_2, u) \end{bmatrix}.$$

Then
$$\beta(f(x,u)) = \varphi_1(\alpha(x), u)$$
but
$$\alpha(f(x,u)) = \varphi(x_1, x_2, u) \neq f_*(\beta(x), u).$$

The binary relation Δ may be given the following interpretation. From (3), if for states $\tilde{x}(t)$ and $\hat{x}(t)$ at time instant t the equality
$$\alpha(\tilde{x}(t)) = \alpha(\hat{x}(t))$$
holds, then at time instant $t+1$ we have
$$\beta(\tilde{x}(t+1)) = \beta(\hat{x}(t+1))$$
independent of the control $u(t)$ applied.

Another interpretation is also possible. One may ask the question. What do we have to know about $x(t)$ to compute $\beta(x(t+1))$ for arbitrary but known $u(t)$? Of course, in the case when all the components of $x(t)$ are known, this is possible. However, in many cases, some of this information is unnecessary and the amount of the necessary information is displayed in function $\alpha(x)$, forming a pair with the function $\beta(x)$.

Obviously, given $\beta(x)$, there exist many functions $\alpha(x)$, forming a pair with $\beta(x)$, i.e. $(\alpha, \beta) \in \Delta$. The most important among them is the maximal function with respect to the relation $\leq$, denoted by $\mathbf{M}(\beta)$. In a similar manner, for a given function $\alpha(x)$, there exist many functions $\beta(x)$, forming a pair with $\alpha(x)$, i.e. $(\alpha, \beta) \in \Delta$. We will denote by $\mathbf{m}(\alpha)$ the minimal function among those functions (with respect to relation $\leq$).

Binary relation Δ is used for definition of the operators $\mathbf{m}$ and $\mathbf{M}$. These operators define the functions $\mathbf{m}(\alpha)$ and $\mathbf{M}(\beta)$ respectively that are supposed to satisfy the conditions formulated below in Definitions 11 and 12.

Definition 11. The function $\mathbf{M}(\beta) \in S_X$ is defined by the following two conditions

(i) $(\mathbf{M}(\beta), \beta) \in \Delta$
(ii) if $(\alpha, \beta) \in \Delta$, then $\alpha \leq \mathbf{M}(\beta)$.

Definition 12. The function $\mathbf{m}(\alpha) \in S_X$ is defined by the following two conditions

(i) $(\alpha, \mathbf{m}(\alpha)) \in \Delta$
(ii) if $(\alpha, \beta) \in \Delta$, then $\mathbf{m}(\alpha) \leq \beta$.

The properties of the operators $\mathbf{M}$ and $\mathbf{m}$ are as follows (see Zhirabok and Shumsky [2008]):

(1) $\alpha \leq \beta \Rightarrow \mathbf{M}(\alpha) \leq \mathbf{M}(\beta)$;
(2) $\alpha \leq \beta \Rightarrow \mathbf{m}(\alpha) \leq \mathbf{m}(\beta)$;
(3) $\mathbf{m}(\alpha \oplus \beta) \cong \mathbf{m}(\alpha) \oplus \mathbf{m}(\beta)$;
(4) $\mathbf{m}(\mathbf{M}(\beta)) \leq \beta$;
(5) $\mathbf{M}(\mathbf{m}(\alpha)) \geq \alpha$;

(6) α is f-invariant function $\Leftrightarrow \mathbf{m}(\alpha) \leq \alpha \Leftrightarrow \alpha \leq \mathbf{M}(\alpha)$

Computation of the operator $\mathbf{m}$. It has proven that the function γ exists that satisfies the condition $(\alpha \times u) \oplus f \cong \gamma(f)$; if f is surjection [2] define $\mathbf{m}(\alpha) \cong \gamma$, see Shumsky [1988]. In this paper we assume that f is surjection.

Note that the latter assumption and the submersivity assumption made in Section 2 are related as follows. The map f is a submersion at a point (x, u) if its differential $df : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ at this point is a surjective linear map.

Because the composition $\gamma(f)$ may be written as γ^+ and $\mathbf{m}(\alpha) \cong \gamma$, one may alternatively write the rule for computation of the operator $\mathbf{m}$ using a backward shift as follows: $\mathbf{m}(\alpha) \cong ((\alpha \times u) \oplus f)^-$.

Computation of the operator $\mathbf{M}$. In the special case when the composite function $\beta(f(x, u))$ can be represented in the form

$$\beta(f(x, u)) = \sum_{i=1}^{d} a_i(x) b_i(u)$$

where $a_1(x)$, $a_2(x)$, …, $a_d(x)$ are arbitrary functions and $b_1(u)$, $b_2(u)$, …, $b_d(u)$ are linearly independent, then

$$\mathbf{M}(\beta) := a_1 \times a_2 \times \cdots \times a_d.$$

For the general case, see Zhirabok and Shumsky [2008].

4. MINIMAL f-INVARIANT FUNCTION

The goal of this section is to find a minimal (containing the maximal number of functionally independent components) vector function $\alpha^0(x)$ such that its forward shift $\alpha^0(f(x, u))$ does not depend on control u. Note that if f is smooth, $\alpha^0(x)$ satisfies the condition

$$\frac{\partial}{\partial u} \alpha^0(f(x, u)) \equiv 0.$$

Though $\alpha^0(x)$ is not unique, all possible choices are equivalent functions.

Note that since the relative degrees of the components of α^0 are two or more, the differentials of $\alpha^0(x)$ span the integrable part of the codistribution $\mathcal{H}_2$ of the one-forms, denoted by $\hat{\mathcal{H}}_2$, i.e. $\hat{\mathcal{H}}_2 = \operatorname{span}_{\mathcal{K}}\{d\alpha^0(x)\}$.

Algorithm 1. (Computation of the minimal f-invariant function α satisfying the condition $\alpha^0 \leq \alpha$). Given α^0, compute recursively, using the formula below

$$\alpha^{i+1} = \alpha^i \oplus \mathbf{m}(\alpha^i) \tag{4}$$

the sequence of nondecreasing functions α^i, $i \geq 1$. By Theorem 1 in Shumsky and Zhirabok [2010c] there exists a finite k such that α^{k+1} is equivalent to α^k, denoted by $\alpha^{k+1} \cong \alpha^k$. Note that the sequence α^k converges at most in n steps. Define $\alpha_* := \alpha^k$, and $\delta^i := \alpha^{i-1}$, for $i = 1, \ldots, n$.

The proposition below demonstrates that δ^k corresponds to the integrable subspace of $\mathcal{H}_{k+1}$, denoted by $\hat{\mathcal{H}}_{k+1}$.

Proposition 13. $\hat{\mathcal{H}}_{k+1} = \operatorname{span}_{\mathcal{K}}\{d\delta^k(x)\}$.

Proof. We give the proof for δ^2 and $\mathcal{H}_3$. The justification for the following steps is completely similar. As shown above $\hat{\mathcal{H}}_2 = \operatorname{span}_{\mathcal{K}}\{d\alpha^0(x)\} = \operatorname{span}_{\mathcal{K}}\{d\delta^1(x)\}$. Next, note that $\mathcal{H}_3$ may be alternatively defined as $\mathcal{H}_3 = \mathcal{H}_2 \cap \mathcal{H}_2^-$ [3]. Note that the integrable subspace of $\mathcal{H}_3$, denoted by $\hat{\mathcal{H}}_3$ may be computed alternatively as $\widehat{\hat{\mathcal{H}}_2 \cap \hat{\mathcal{H}}_2^-}$. Indeed, if the exact one-form $d\zeta \in \hat{\mathcal{H}}_3$, one necessarily has $d\zeta \in \mathcal{H}_2$, $d\zeta \in \mathcal{H}_2^-$, and so, also into their intersection. Since $d\zeta$ is exact, $d\zeta \in \hat{\mathcal{H}}_2$ and $d\zeta \in \hat{\mathcal{H}}_2^-$, and therefore, also into their intersection as well as into the integrable subspace of the intersection. To show the converse, note that $\mathcal{H}_3 = \mathcal{H}_2 \cap \mathcal{H}_2^-$ necessarily yields $\hat{\mathcal{H}}_2 \cap \hat{\mathcal{H}}_2^- \subset \mathcal{H}_3$. Next, though $\operatorname{span}_{\mathcal{K}}\{d\mathbf{m}(\delta^1)\} \neq \hat{\mathcal{H}}_2^-$ completely, these two distributions differ by a single basis element. Since this basis element is missing in $\hat{\mathcal{H}}_2$, it does not affect the intersection. In particular, by definition of the operator $\mathbf{m}$, $\delta^1 \times u \leq \mathbf{m}(\delta^1) \circ f = \mathbf{m}(\delta^1)^+$, and therefore $(\delta^1)^- \times u^- \leq \mathbf{m}(\delta^1)$. By definition, the function $(\delta^1)^-$ contains the variable $\tilde{x}$ whereas $\mathbf{m}(\delta^1)$ is free from this variable. Therefore, because $\mathbf{m}(\delta^1)$ is the minimal function, satisfying this inequality,

$$\begin{aligned} \operatorname{span}_{\mathcal{K}}\{d\mathbf{m}(\delta^1)\} &= \operatorname{span}_{\mathcal{K}}\{d(\delta^1)^-\} - \operatorname{span}_{\mathcal{K}}\{d\tilde{x}\} \\ &\quad + \operatorname{span}_{\mathcal{K}}\{du^-\} \\ &= \hat{\mathcal{H}}_2^- - \operatorname{span}_{\mathcal{K}}\{d\tilde{x}\} + \operatorname{span}_{\mathcal{K}}\{du^-\}. \end{aligned}$$

Then

$$\begin{aligned} \operatorname{span}_{\mathcal{K}}\{d(\delta^2)\} &= \operatorname{span}_{\mathcal{K}}\{d(\delta^1 \oplus \mathbf{m}(\delta^1))\} \\ &= \operatorname{span}_{\mathcal{K}}\{d\delta^1\} \cap \operatorname{span}_{\mathcal{K}}\{d\mathbf{m}(\delta^1)\} \\ &= \hat{\mathcal{H}}_2 \cap (\hat{\mathcal{H}}_2^- + \operatorname{span}_{\mathcal{K}}\{du^-\}) \end{aligned}$$

that corresponds to $\hat{\mathcal{H}}_3 = \hat{\mathcal{H}}_2 \cap \hat{\mathcal{H}}_2^-$. ■

Examples 14 and 15 below illustrate the Proposition 13. Namely, note that in the span of $\hat{\mathcal{H}}_2^-$ the differential of the control variable du^- is missing but $\hat{\mathcal{H}}_2^-$ contains instead the element of $d\tilde{x}^-$ whereas $\tilde{x}^-$ is missing in $\mathbf{m}(\delta^1)$. Moreover, $\delta^1 \oplus \mathbf{m}(\delta^1)$ corresponds to $\hat{\mathcal{H}}_2 \cap \hat{\mathcal{H}}_2^-$.

Example 14. Consider the system

$$\begin{aligned} x_1^+ &= x_1 + x_3 \\ x_2^+ &= x_2 + x_5 \\ x_3^+ &= u \\ x_4^+ &= x_3 x_4 \\ x_5^+ &= x_1 \end{aligned}$$

Note that for this example (if we take $\tilde{x} = x_5$)

$$\begin{aligned} x_1^- &= x_5 \\ x_2^- &= x_2 - \tilde{x}^- \\ x_3^- &= x_1 - x_5 \\ x_4^- &= x_4/(x_1 - x_5) \\ x_5^- &= \tilde{x}^- \end{aligned}$$

and $u^- = x_3$.

Compute

$$\delta^1 = [x_1, x_2, x_4, x_5]^T$$

and

$$\begin{aligned} \mathbf{m}(\delta^1) &= [[x_1, x_2, x_4, x_5, u] \oplus [x_1 + x_3, x_2 + x_5, u, x_3 x_4, x_1]]^- \\ &= [x_2 + x_5, u, x_1, x_4]^- = [x_2, x_3, x_5, \frac{x_4}{x_1 - x_5}] \end{aligned}$$

[2] For non-surjective f the formula is more complicated.

[3] Note that the application of the backward shift to the codistribution has to be understood componentwise.

Note that δ^1 corresponds to $\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_4, \mathrm{d}x_5\}$, which is integrable and $\mathbf{m}(\delta^1)$ corresponds to the $\mathcal{H}_2^-$, where

$$\begin{aligned}\mathcal{H}_2^- &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1^-, \mathrm{d}x_2^-, \mathrm{d}x_4^-, \mathrm{d}x_5^-\}\\ &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1^-, \mathrm{d}x_2^- + \mathrm{d}x_5^-, \mathrm{d}x_4^-, \mathrm{d}x_5^-\}\\ &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_5, \mathrm{d}x_2, \mathrm{d}\left(\frac{x_4}{x_1 - x_5}\right), \mathrm{d}\tilde{x}^-\}\end{aligned}.$$

The only difference between $\mathbf{m}(\delta^1)$ and $\mathcal{H}_2^-$ is that whereas $\mathbf{m}(\delta^1)$ contains $x_3 = u^-$, $\mathcal{H}_2^-$ contains $\mathrm{d}\tilde{x}^- = \mathrm{d}x_5$. All the other components coincide.

Furthermore, compute

$$\delta^2 = \delta^1 \oplus \mathbf{m}(\delta^1) = \left[x_2, x_5, \frac{x_4}{x_1 - x_5}\right],$$

and

$$\begin{aligned}\mathbf{m}(\delta^2) &= \left[\left[x_2, x_5, \frac{x_4}{x_1 - x_5}, u\right]\right.\\ &\left.\oplus\ [x_1 + x_3, x_2 + x_5, u, x_3x_4, x_1]\right]^-\\ &= [x_2 + x_5, u]^- = [x_2, x_3].\end{aligned}$$

Note that δ^2 corresponds to (integrable)

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}\left(\frac{x_4}{x_1 - x_5}\right)\right\}$$

In a similar manner, compute

$$\delta^3 = \delta^2 \oplus \mathbf{m}(\delta^2) = x_2$$

and

$$\begin{aligned}\mathbf{m}(\delta^3) &= [[x_2, u] \oplus [x_1 + x_3, x_2 + x_5, u, x_3x_4, x_1]]^-\\ &= u^- = x_3\end{aligned}$$

Note that δ^3 corresponds to the integrable subspace of $\mathcal{H}_4$.

Therefore

$$\delta^4 = \delta^3 \oplus \mathbf{m}(\delta_3) \cong \mathrm{const}.$$

Finally, note that $\delta^4 \cong$ const corresponds to $\mathcal{H}_\infty$ being trivial, $\mathcal{H}_\infty = \{0\}$.

Example 15. Consider the system

$$\begin{aligned}x_1^+ &= x_1 + x_3\\ x_2^+ &= x_2\\ x_3^+ &= u\\ x_4^+ &= x_3x_4\\ x_5^+ &= x_1\end{aligned}$$

Compute

$$\delta^1 = [x_1, x_2, x_4, x_5]^T$$

and

$$\begin{aligned}\mathbf{m}(\delta^1) &= [[x_1, x_2, x_4, x_5, u] \oplus [x_1 + x_3, x_2, u, x_3x_4, x_1]]^-\\ &= [x_2 + x_5, u, x_1, x_4]^- = [x_2, x_3, x_5, \frac{x_4}{x_1 - x_5}]\end{aligned}$$

Note that δ^1 corresponds to $\mathcal{H}_2 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_4, \mathrm{d}x_5\}$. Furthermore,

$$\delta^2 = \delta^1 \oplus \mathbf{m}(\delta^1) = \left[x_2, x_5, \frac{x_4}{x_1 - x_5}\right],$$

and

$$\begin{aligned}\mathbf{m}(\delta^2) &= \left[\left[x_2, x_5, \frac{x_4}{x_1 - x_5}, u\right] \oplus [x_1 + x_3, x_2, u, x_3x_4, x_1]\right]^-\\ &= [x_2, u]^- = [x_2, x_3].\end{aligned}$$

Note that δ^2 corresponds to

$$\mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\left\{\mathrm{d}x_2, \mathrm{d}x_5, \mathrm{d}\left[\frac{x_4}{x_1 - x_5}\right]\right\}.$$

In a similar manner, compute

$$\delta^3 = \delta^2 \oplus \mathbf{m}(\delta^2) = x_2$$

and

$$\begin{aligned}\mathbf{m}(\delta^3) &= [[x_2, u] \oplus [x_1 + x_3, x_2, u, x_3x_4, x_1]]^-\\ &= u^- = [x_2, u]^- = [x_2, x_3].\end{aligned}$$

Note that δ^3 corresponds to the integrable subspace of $\mathcal{H}_4$, i.e. $\hat{\mathcal{H}}_4 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2\}$.

Therefore

$$\delta^4 = \delta^3 \oplus \mathbf{m}(\delta_3) = x_2 = \delta^3$$

Finally, note that $\delta^4 \cong \delta^3 = x_2$ corresponds to the fact that $\mathcal{H}_\infty = \mathcal{H}_5 = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_2\}$.

5. ACCESSIBILITY

Note that accessibility is a necessary condition for static state feedback linearizability. Therefore, we recall below the accessibility definition and condition formulated in terms of codistribution.

Following the notation in Jakubczyk and Sontag [1990] we denote by $A_k(x)$ the set of points reachable from x in k forward time steps using arbitrary sequences of controls $\mathbf{u} = (u(0), \ldots, u(k-1)) \in (R^m)^k$, and by $A(x)$ the set of points reachable from x in any number of forward steps using arbitrary sequences of controls. That is,

$$A(x) = \bigcup_{k \geq 0} A_k(x).$$

The system is said to be forward accessible from x if its reachable set $A(x)$ has non-empty interior. A generic notion of accessibility has been derived from this pointwise definition in Albertini and Sontag [1993].

Definition 16. The system (1) is said to be (forward) accessible if its reachable set $A(x)$ has a non-empty interior in $\mathbb{R}^n$ for almost all $x \in \mathbb{R}^n$.

Proposition 17. Aranda-Bricaire et al. [1996] The system (1) is accessible iff $\mathcal{H}_\infty = \{0\}$.

Proposition 18. The following statements for system (1) are equivalent

(i) $\mathcal{H}_\infty \neq \{0\}$

(ii) For some k, $\delta^{k-1} \cong \delta^k \neq \text{const}$.

Proof. Suppose $\mathcal{H}_\infty \neq 0$. This means that f-invariant function β_* exists such that $\alpha^0 \leq \beta_*$. Due to the 6th property of the operators $\mathbf{m}$ and $\mathbf{M}$, $\mathbf{m}(\beta_*) \leq \beta_*$. The inequality $\alpha^0 \leq \beta_*$ implies $\mathbf{m}(\alpha^0) \leq \mathbf{m}(\beta_*)$ which together with the above inequality gives $\mathbf{m}(\alpha^0) \leq \beta_*$. By analogy one may prove that for all i, $\mathbf{m}^i(\alpha^0) \leq \beta_*$. It follows from (4) that $\delta^i \leq \beta_*$ for all i. Due to (4) $\delta^1 \leq \delta^2 \leq \ldots$ and since this sequence is bounded, then $\delta^{k-1} \cong \delta^k \neq \text{const}$ for some k.

Suppose that $\delta^{k-1} \cong \delta^k \neq \text{const}$ for some k. This means that the function $\alpha_* \cong \delta^k$ is f-invariant such that $\alpha^0 \leq \alpha_*$, i.e. the system (1) may be decomposed into the form containing an autonomous subsystem. The latter means that that the system (1) is nonaccessible since it admits an autonomous variable, see Aranda-Bricaire et al. [1996] and therefore $\mathcal{H}_\infty \neq \{0\}$. ■

6. FEEDBACK LINEARIZATION

Definition 19. A regular static state feedback $u = \alpha(x, v)$ is a mapping $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that generically $[\partial\alpha(\cdot)/\partial v] \neq 0$.

System (1) is said to be static state feedback linearizable if, generically, there exist

(i) a state coordinate transformation $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and
(ii) a regular static state feedback of the form $u = \alpha(x, v)$, such that, in the new coordinates $z = \varphi(x)$, the compensated system reads $z^+ = Az + Bv$, where the pair (A, B) is in Brunovsky canonical form.

Theorem 20. Aranda-Bricaire et al. [1996] System (1) is static state feedback linearizable if and only if

(i) $\mathcal{H}_k$ is completely integrable for all $k = 1, \ldots, n$,
(ii) $\mathcal{H}_\infty := \mathcal{H}_{n+1} = \{0\}$.

Theorem 21 below suggests an alternative solution to the static state feedback linearization problem. Consider a special form of the system (1), the so-called controller canonical form, see Kotta [2005]:

$$\begin{aligned} z_1^+ &= z_2, \\ z_2^+ &= z_3, \\ &\vdots \\ z_{n-1}^+ &= z_n \\ z_n^+ &= \psi(z, u). \end{aligned} \tag{5}$$

The goal of Theorem 21 below is to find out under which conditions formulated in terms of the algebra of functions the system (1) can be transformed into the form (5) using the state coordinate transformation $z = \varphi(x)$. From this form a regular static state feedback may be easily found by defining $v = \psi(z, u)$, in order to solve the feedback linearization problem.

Note that by the results of Kotta [2005] the conditions to transform the system (1) into the form (5) coincide with those of Theorem 20.

Theorem 21. The system (1) can be transformed into the form (5) iff $\delta^i \neq \text{const}$, for $i = 1, \ldots, n-1$, but $\delta^n = \text{const}$ where $\delta^i = \alpha^{i-1}$ for $i = 1, 2, \ldots, n$.

Proof. Sufficiency. Define $\varphi_1 := \delta^{n-1}$, $\varphi_{i+1} = \mathbf{M}(\varphi_i)$, $i = 1, 2, \ldots, n-1$ and $z_i = \varphi_i(x)$, $i = 1, 2, \ldots, n$. According to the definition of the operator $\mathbf{M}$, one has $(\mathbf{M}(\varphi_i), \varphi_i) \in \Delta$ that together with $\varphi_{i+1} = \mathbf{M}(\varphi_i)$ gives $(\varphi_{i+1}, \varphi_i) \in \Delta$. By definition of the binary relation Δ, there exists a function f_{*i} such that

$$\varphi_i(f(x, u)) = f_{*i}(\varphi_{i+1}(x), u),$$

for all $(x, u) \in X \times U$ and $i = 1, 2, \ldots, n-1$. It follows from (4) that $\alpha^0 = \delta^1 \leq \delta^2 \leq \ldots \leq \delta^{n-1}$. Since $\delta^{n-1} = \delta^{n-2} \oplus \mathbf{m}(\delta^{n-2})$, we have $\delta^{n-1} \geq \mathbf{m}(\delta^{n-2})$, and therefore, by the 1st and 5th properties of the operators $\mathbf{m}$ and $\mathbf{M}$ the following holds

$$\begin{aligned} \varphi_2 := \mathbf{M}(\varphi_1) = \mathbf{M}(\delta^{n-1}) &\geq \mathbf{M}(\mathbf{m}(\delta^{n-2})) \\ &\geq \delta^{n-2} \geq \alpha^0. \end{aligned}$$

In a similar manner, one may obtain the inequalities

$$\begin{aligned} \varphi_3 &= \mathbf{M}(\varphi_2) \geq \delta^{n-3} \geq \alpha^0, \\ &\vdots \\ \varphi_{n-1} &= \mathbf{M}(\varphi_{n-2}) \geq \delta^1 \geq \alpha^0. \end{aligned}$$

The definition of the function α^0 and the inequalities $\varphi_i \geq \alpha^0$, $i = 1, 2, \ldots, n-1$, yield

$$\frac{\partial}{\partial u}\varphi_i(f(x, u)) = \frac{\partial}{\partial u} f_{*i}(\varphi_{i+1}(x), u) = 0,$$

therefore the function f_{*i} does not depend on u and

$$\varphi_i(f(x, u)) = f_{*i}(\varphi_{i+1}(x)).$$

The last equality and the rule for computation of the operator $\mathbf{M}$ yield $\mathbf{M}(\varphi_i) = f_{*i}(\varphi_{i+1})$. Since $\varphi_{i+1} = \mathbf{M}(\varphi_i)$, one may take $f_{*i}(\varphi_{i+1}) = \varphi_{i+1}$, $i = 1, 2, \ldots, n-1$. Then

$$z_i^+ = \varphi_i(f(x, u)) = \varphi_{i+1}(x) = z_{i+1}$$

for $i = 1, 2, \ldots, n-1$. Since the function φ_n does not satisfy the condition $\varphi_n \geq \alpha^0$, the equation for the variable z_n has the general form:

$$z_n^+ = \varphi_n(f(x, u)) = \psi(z, u).$$

Necessity. Suppose the system (1) can be transformed into the canonical form (5), i.e. there exists the state transformation $\varphi : X \to Z$ such that for $i = 1, \ldots, n-1$

$$z_i^+ = \varphi_i(x^+) = \varphi_i(f(x, u)) = \varphi_{i+1}(x) = z_{i+1}, \tag{6}$$

and

$$z_n^+ = \varphi_n(x^+) = \psi(z, u). \tag{7}$$

The equality $\varphi_i(f(x, u)) = \varphi_{i+1}(x)$ in (6) and a definition of the binary relation Δ yields the inclusion $(\varphi_{i+1}, \varphi_i) \in \Delta$, for $i = 1, 2, \ldots, n-1$. It is obvious that

$$\frac{\partial}{\partial u}\varphi_i(f(x, u)) = \frac{\partial}{\partial u}\varphi_{i+1}(x) = 0,$$

therefore, by the definition of the function α^0, $\alpha^0 \leq \varphi_i$ for $i = 1, 2, \ldots, n-1$. By the 2nd property of the operator $\mathbf{m}$, this inequality implies $\mathbf{m}^j(\alpha^0) \leq \mathbf{m}^j(\varphi_i)$ for all $j \geq 1$, and $i = 1, 2, \ldots, n-1$.

Consider the inclusion $(\varphi_2, \varphi_1) \in \Delta$ which is by Definition 12 equivalent to the inequality $\mathbf{m}(\varphi_2) \leq \varphi_1$ that together with $\mathbf{m}(\alpha^0) \leq \mathbf{m}(\varphi_2)$ gives $\mathbf{m}(\alpha^0) \leq \varphi_1$. In the similar manner the inclusion $(\varphi_3, \varphi_2) \in \Delta$ is equivalent to the inequality $\mathbf{m}(\varphi_3) \leq \varphi_2$ which implies $\mathbf{m}^2(\varphi_3) \leq \mathbf{m}(\varphi_2)$. Since $\mathbf{m}^2(\alpha^0) \leq \mathbf{m}^2(\varphi_3)$ and $\mathbf{m}(\varphi_2) \leq \varphi_1$, one obtains from these three inequalities $\mathbf{m}^2(\alpha^0) \leq \varphi_1$. Analogously, it can be proved that $\mathbf{m}^i(\alpha^0) \leq \varphi_1$, $i = 1, 2, \ldots, n-2$. By definition of the operation $\oplus$, these inequalities are equivalent to the single inequality

$$\alpha^0 \oplus \mathbf{m}(\alpha^0) \oplus \ldots \oplus \mathbf{m}^i(\alpha^0) \leq \varphi_1, \quad i = 1, 2, \ldots n-2. \tag{8}$$

It follows from the definition of the functions δ^i, $i = 1, 2, \ldots, n-1$, and the 3rd property of the operator $\mathbf{m}$ that

$$\begin{aligned} \delta^3 &= \delta^2 \oplus \mathbf{m}(\delta^2) \\ &= (\delta^1 \oplus \mathbf{m}(\delta^1)) \oplus \mathbf{m}(\delta^1 \oplus \mathbf{m}(\delta^1)) \\ &\cong \alpha^0 \oplus \mathbf{m}(\alpha^0) \oplus \mathbf{m}^2(\alpha^0). \end{aligned}$$

In general, it can be proved that

$$\delta^i \cong \alpha^0 \oplus \mathbf{m}(\alpha^0) \oplus \ldots \oplus \mathbf{m}^{i-1}(\alpha^0), \quad i = 1, 2, \ldots, n-1.$$

Since $\varphi_1 \neq$ const, then due to (8), $\delta^i \neq$ const, for $i = 1, 2, \ldots, n-1$.

Suppose now contrarily to the claim of the theorem that $\delta^n = \alpha^0 \oplus \mathbf{m}(\alpha^0) \oplus \ldots \oplus \mathbf{m}^n(\alpha^0) \neq$ const. By analogy with the proof of sufficiency part of the theorem, it can be shown that $\dfrac{\partial}{\partial u}\varphi_n(f(x,u)) = \dfrac{\partial}{\partial u}\psi(z,u) = 0$, i.e. the right-hand side of the equation $z_n^+ = \varphi_n(f(x,u))$ does not depend on the control u that contradicts the equation (5). ■

Proposition 22. The following two conditions for system (1) are equivalent

(i) $\mathcal{H}_\infty = \{0\}$ and $\mathcal{H}_k$, for $k = 1, \ldots, n$ are completely integrable
(ii) $\delta^i \neq$ const for $i = 1, \ldots, n-1$ and $\delta^n =$ const.

Proof. (ii) $\to$ (i) Suppose that $\delta^i \neq$ const for $i = 1, 2..., n-1$, and $\delta^n =$ const. According to Theorem 21, in this case the system (1) can be transformed into the controller canonical form (5) and therefore, it is accessible. Then by Proposition 17, $\mathcal{H}_\infty = \{0\}$. Moreover, by the results of Kotta [2005], $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are completely integrable.

(i) $\to$ (ii) Consider a sequence of functions $\alpha^0 := \delta^1 \leq \delta^2 \leq \ldots$. Since the sequence converges at most in n steps, for some k, $\delta^{k+1} = \delta^k$. If $\delta^k \neq$ const, then it follows from Proposition 18 that $\mathcal{H}_\infty \neq \{0\}$ which contradicts the condition (i). Therefore for some k, $\delta^k =$ const. According to Theorem 20, the system (1) is static state feedback linearizable, and therefore, due to the structure of the Brunovsky canonical form, $k = n$. ■

Remark 1. If $\delta^{k-1} \cong \delta^k \neq$ const holds for some k, then $\varphi_1 \cong \mathbf{M}(\varphi_1)$. The latter means that the function φ_1 is f-invariant and the variable $z_1 := \varphi_1(x)$ satisfies the equation $z_1^+ = z_1$, and the system (1) is not transformable into the canonical form (5).

Remark 2. If the system is not transformable into the canonical form (5) one may alternatively say that function φ_1 in Remark 1 is an autonomous variable for system (1), and therefore, the system (1) is non-accessible.

7. EXAMPLES

Example 23. (Continuation of Example 14)

Since for this example $n = 5$, but already $\delta^4 \cong$ const, the system admits only partial linearization.

Example 24. (Continuation of Example 15) Since $\delta^3 = \delta^4 = x_2$, the system is not accessible.

Example 25. Consider the system

$$\begin{aligned} x_1^+ &= x_1(x_3^2+1)^2 \\ x_2^+ &= x_2(x_3^2+1)^3 \\ x_3^+ &= x_3 + u \end{aligned} \tag{9}$$

Compute first according to Aranda-Bricaire et al. [1996]

$$\mathcal{H}_\infty = \mathcal{H}_3 = \mathrm{span}_{\mathcal{K}}\{3x_2\mathrm{d}x_1 - 2x_1\mathrm{d}x_2\};$$

therefore the system (9) admits an autonomous variable x_1^3/x_2^2.

Next, using the tools of the algebra of functions one may compute

$$\delta^1 := \alpha^0(x) = [x_1, x_2]^T.$$

Furthermore, by (4) [4] [5],

$$\delta^2(x) = \delta^1(x) \oplus \mathbf{m}(\delta^1(x)) = [x_1, x_2]^T \oplus \frac{x_1^3}{x_2^2} = \frac{x_1^3}{x_2^2}.$$

Since

$$\mathbf{m}\left(\frac{x_1^3}{x_2^2}\right) = \frac{x_1^3}{x_2^2},$$

we get

$$\delta^3(x) \cong \delta^2(x) \neq \text{const}.$$

Example 26. Consider the system with non-smooth state transition map $f(x,u)$,

$$\begin{aligned} x_1^+ &= x_2^2 u \\ x_2^+ &= x_1 \mathrm{sign}\ x_2 \\ x_3^+ &= u \end{aligned}$$

Compute

$$\delta^1 := \alpha^0 = \left[\frac{x_1}{x_3}, x_2\right]$$

and

$$\begin{aligned} \mathbf{m}(\delta^1) &= [[\delta^1, u] \oplus f(x,u)]^- \\ &= \left[\left[\frac{x_1}{x_3}, x_2, u\right] \oplus [x_2^2 u, x_1 \mathrm{sign}\ x_2, u]\right]^- \\ &= [x_2^2 u, u]^- = [x_1, x_3]. \end{aligned}$$

So,

$$\delta^2 = \delta^1 \oplus \mathbf{m}(\delta^1) = \left[\frac{x_1}{x_3}, x_2\right] \oplus [x_1, x_3] = \frac{x_1}{x_3}$$

Furthermore, compute

$$\begin{aligned} \mathbf{m}(\delta^2) &= [[\delta^2, u] \oplus f(x,u)]^- \\ &= \left[\left[\frac{x_1}{x_3}, u\right] \oplus [x_2^2 u, x_1 \mathrm{sign}\ x_2, u]\right]^- \\ &= u^- = x_3 \end{aligned}$$

[4] $\mathbf{m}(\zeta)$ is a function of x at the time instant $t+1$ which can be computed if ζ is known at time instant t.

[5] $\delta^2(x)$ is a function of x which can be computed both from $\delta^1(x)$ and $\mathbf{m}(\delta^1(x))$.

and

$$\delta^3 = \delta^2 \oplus \mathbf{m}(\delta^2) = \frac{x_1}{x_3} \oplus x_3 = \text{const}.$$

Therefore, define

$$z_1 := \delta^2 = \frac{x_1}{x_3}$$

$$z_2 := z_1^+ = \mathbf{M}(\delta^2) = x_2^2$$

$$z_3 := z_2^+ = \mathbf{M}^2(\delta^2) = \mathbf{M}(x_2^2) = x_1^2(\text{sign } x_2)^2$$

and find the state equations

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 \\ z_3^+ &= z_2^2 u^2 \left(\text{sign}\sqrt{z_3}\right)^2 \end{aligned}$$

Example 27. Consider

$$\begin{aligned} x_1^+ &= ux_1 \\ x_2^+ &= x_1 x_3 \\ x_3^+ &= x_3 \end{aligned}$$

Compute

$$\delta^1 := \alpha^0 = [x_2, x_3]^T$$

and

$$\begin{aligned} \mathbf{m}(\delta^1) &= [[x_2, x_3, u] \oplus [ux_1, x_1x_3, x_3]]^- \\ &= x_3^- = x_3. \end{aligned}$$

Furthermore,

$$\delta^2 \cong \delta^1 \oplus \mathbf{m}(\delta^1) = x_3.$$

Obviously, $\delta^3 \cong \delta^2 \neq$ const. The system admits an autonomous element x_3. The system is neither accessible nor linearizable via static state feedback.

Example 28. Consider the system

$$\begin{aligned} x_1^+ &= \zeta(x_2)u \\ x_2^+ &= x_1 x_2 \\ x_3^+ &= u \end{aligned}$$

where ζ is an invertible analytic function, and compute

$$\mathcal{H}_2 = \text{span}_{\mathcal{K}} \left\{ \mathrm{d}\left(\frac{x_1}{x_3}\right), \mathrm{d}x_2 \right\}$$

and

$$\mathcal{H}_3 = \text{span}_{\mathcal{K}} \left\{ \mathrm{d}\left(\frac{x_1}{x_3}\right) \right\}, \quad \mathcal{H}_3 = \{0\}.$$

Define the new state variables

$$\begin{aligned} z_1 &= \frac{x_3}{x_3} \\ z_2 &= \left(\frac{x_1}{x_3}\right)^+ = \zeta(x_2) \\ z_3 &= \zeta^+(x_2) = \zeta(x_2^+) = \zeta(x_1, x_2) \end{aligned} \tag{10}$$

and so the state equations in the controller canonical form are

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 \\ z_3^+ &= \zeta(\zeta^{-1}(z_3)z_2 u). \end{aligned} \tag{11}$$

Note that using the results of Theorem 21, one does not have to assume $\zeta(x_2)$ to be analytic. Compute

$$\delta^1 := \alpha^0 = [x_1/x_3, x_2]$$

and

$$\begin{aligned} \mathbf{m}(\delta^1) &= [[\delta^1, u] \oplus f(x, u)]^- \\ &= [[x_1/x_3, x_2, u] \oplus [\zeta(x_2)u, x_1x_2, u]]^- \\ &= [\zeta(x_2)u, u]^- = [x_1, x_3] \end{aligned}$$

So, $\delta^2 = \delta_1 \oplus \mathbf{m}(\delta^1) = x_1/x_3$. Then

$$\begin{aligned} \mathbf{m}(\delta^2) &= [[\delta^2, u] \oplus f(x, u)]^- \\ &= [[x_1/x_3, u] \oplus [\zeta(x_2)u, x_1x_2, u]]^- \\ &= u^- = x_3 \end{aligned}$$

and $\delta^3 = \delta^2 \oplus \mathbf{m}(\delta^2) = \text{const}$.

Therefore, define $z_1 = \delta^2 = x_1/x_3$, $z_2 = \mathbf{M}(\delta^2) = \zeta(x_2)$, $z_3 = \mathbf{M}^2(\delta^2) = \zeta(x_1x_2)$ that agrees with (10) yielding (11).

Example 29. Consider the system

$$\begin{aligned} x_1^+ &= x_3, \\ x_2^+ &= \text{sign}(x_3) + x_1, \\ x_3^+ &= ux_1. \end{aligned} \tag{12}$$

Note that this system cannot be studied using the results of Theorem 21 since f in (12) is not a smooth function. Compute

$$\delta^1 = [x_1, x_2],$$

and

$$\begin{aligned} \mathbf{m}(\delta^1) &= [[x_1, x_2, u] \oplus [x_3, sign(x_3) + x_1, ux_1]]^- \\ &= [x_1, ux_1]^- = [x_2 - sign(x_1), x_3]. \end{aligned}$$

Therefore,

$$\delta^2 = x_2 - sign(x_1).$$

Next, compute $\mathbf{m}(\delta^2) = \text{const}$, yielding $\delta^3 = \text{const}$. By the results of Theorem 21 (see the proof), we have

$$\begin{aligned} z_1 &:= \delta^{n-1} = \delta^2 = x_2 - sign(x_1), \\ z_2 &:= z_1^+ = x_1, \\ z_3 &:= z_2^+ = x_3, \end{aligned}$$

yielding the equations in the controller canonical form

$$\begin{aligned} z_1^+ &= z_2, \\ z_2^+ &= z_3, \\ z_3^+ &= uz_2 \end{aligned}$$

that are easily linearizable by using the feedback $u = z_2/v$.

8. CONCLUSIONS

For the discrete-time nonlinear SISO control systems the problems of accessibility and static state feedback linearizability have been readdressed in terms of the new tools, called the algebra of functions. Unlike the differential geometric methods the new tools allow to study the non-smooth systems. The new results are compared to the existing ones and the relationship is demonstrated on the numerous examples.

The extension of the results for the continuous-time case is not immediate, since the inequality $\delta^{k-1} \geq \mathbf{m}(\delta^{k-2})$ in the continuous-time case, unlike the discrete-time case, does not yield the inequality $\mathbf{M}(\delta^{k-1}) \geq \mathbf{M}(\mathbf{m}(\delta^{k-2}))$ which was used in the proof.

REFERENCES

F. Albertini and E.D. Sontag. Discrete-time transitivity and accessibility: analytic systems. *SIAM J. Contr. Optim.*, 31:1599–1622, 1993.

E. Aranda-Bricaire, Ü. Kotta, and C.H. Moog. Linearization of discrete-time systems. *SIAM J. Contr. Optim.*, 34(6):1999–2023, 1996.

G. Conte, C.H. Moog, and A.M. Perdon. *Algebraic Methods for Nonlinear Control Systems. Theory and Applications.* Springer, 2007.

J.W. Grizzle. Feedback linearization of discrete-time systems. In *Lecture Notes in Control and Information Sciences*, volume 83, pages 273–281. Springer, New York, 1986.

J.W. Grizzle. A linear algebraic framework for the analysis of discrete-time nonlinear systems. *SIAM J. Contr. Optim.*, 31:1026–1044, 1993.

J. Hartmanis and R. Stearns. *The Algebraic Structure Theory of Sequential Machines.* Prentice-Hall, New York, 1966.

B. Jakubczyk. Feedback linearization of discrete-time systems. *Systems and Control Letters*, 9:411–416, 1987.

B. Jakubczyk and E.D. Sontag. Controllability of nonlinear discrete-time systems: a lie algebraic approach. *SIAM J. Contr. Optim.*, 28:1–33, 1990.

G. Jayaraman and H.J Chizek. Feedback linearization of discrete-time systems. In *Proc. of the 32nd IEEE Conference on Decision and Control*, pages 2372–2977. San Antonion, Texas, 1993.

Ü. Kotta. Controller and controllability canonical forms for discrete-time nonlinear systems. *Proceedings of the Estonian Academy of Sciences.Physics.Mathematics*, 54 (1):55–62, 2005.

K. Nam. Linearization of discrete-time nonlinear systems and a canonical structure. *IEEE Trans. Autom. Control*, 34:119–122, 1989.

H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems.* Springer, New York, 1990.

A. Shumsky and A. Zhirabok. Method of accommodation to the defect in finite automata. *Automation and Remote Control*, 71(5):837–846, 2010a.

A. Shumsky and A. Zhirabok. Unified approach to the problem of fault diagnosis. In *Proc. Conf. on Control and Fault Tolerant Systems*, pages 450–55. Nice, France, 2010b.

A.Ye Shumsky. Model of faults for discrete systems and its application to functional diagnosis problem. *Engineering Simulation*, 12(4):56–61, 1988.

A.Ye. Shumsky. Full decoupling via feedback in nonlinear descriptor systems: algebraic approach. In *Proc. of the 6th IFAC Symposium on Robust Control Design*, pages 261–266. Haifa, Israel, 2009.

A.Ye. Shumsky and A.N. Zhirabok. Unified approach to the problem of full decoupling via output feedback. *European Journal of Control*, 16(4):313–325, 2010c.

C. Simões and H. Nijmeijer. Nonsmooth stabilizability and feedback linearization of discrete-time systems. *International Journal of Robust and Nonlinear Control*, 6: 171–188, 1996.

A.N. Zhirabok and A.Ye. Shumsky. *The algebraic methods for analysis of nonlinear dynamic systems (In Russian).* Dalnauka, Vladivostok, 2008.