Slovak University of Technology in Bratislava
Institute of Information Engineering, Automation, and Mathematics

PROCEEDINGS

of the 18$^{\text{th}}$ International Conference on Process Control

Hotel Titris, Tatranská Lomnica, Slovakia, June 14 – 17, 2011

ISBN 978-80-227-3517-9

http://www.kirp.chtf.stuba.sk/pc11

Editors: M. Fikar and M. Kvasnica

Dušek, F., Honc, D., Rozsival, P.: Mathematical Model of Differentially Steered Mobile Robot, Editors: Fikar, M., Kvasnica, M., In *Proceedings of the 18th International Conference on Process Control*, Tatranská Lomnica, Slovakia, 221–229, 2011.

Full paper online: http://www.kirp.chtf.stuba.sk/pc11/data/abstracts/023.html

Mathematical model of differentially steered mobile robot

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Abstract: Paper deals with dynamic mathematical model of an ideal differentially steered drive system (mobile robot) planar motion. The aim is to create model that describes trajectory of a robot's arbitrary point. The trajectory depends on supply voltage of both drive motors. Selected point trajectory recomputation to trajectories of wheels contact points with plane of motion is a part of the model, too. The dynamic behaviour of engines and chassis, form of coupling between engines and wheels and basic geometric dimensions are taken into account. The dynamic model will be used for design and verification of a robot's motion control in MATLAB / SIMULINK simulation environment.

1. INTRODUCTION

Paper deals with dynamic model of an ideal mobile robot with differentially steered drive system and planar motion. Single-axle chassis or caterpillar chassis is mostly used in case of small mobile robots (Novák 2005). Caster wheel is added to single-axle to ensure stability. This solution together with independent wheel actuation allows excellent mobility on the contrary to a classic chassis – see commercially available robot in Fig. 1. Derived mathematical model comes from lay-out, nominal geometric dimensions and other features of that robot with view of ideal behaviour of individual components and some simplifying assumptions. The aim is to create model based on forces caused by engine moments of independent wheel drives. Model will consist of dynamic behaviour description of chassis and in series connected DC motors. Presented motion model based on centre of mass (primary element) dynamics is different from models reflecting kinematics only and commonly used in literature – published e.g. in (Stengel 2010) or (Lucas 2010). Standard models describe robot's trajectory time evaluation depending up to known wheel speed (information from wheel speed sensors) and chassis geometry - odometry – published e.g. in (Winkler 2010). Our model extends standard model with dynamic part describing wheel speed dependency on motor supply voltage by respecting dynamics, construction, geometry and other parameters of chassis and motors.

Motor supply voltage actuating the wheel causes driving torque and thereby wheel rotation. Inertial and resistance forces act against driving torque. Both driving torques influence each other because of these forces. Planar curvilinear motion of the robot is result of various time variant wheel rotation speeds.

Planar curvilinear motion can be decomposed to a sum of linear motion (translation) and rotation motion. Forces balance is starting point for the derivation of motion equations. If F is actual force acting to a mass point with weight m and with distance r from axis of rotation then it holds for general curvilinear motion that vector sum of all forces acting to a selected point is zero - see literature (Horák et al. 1976)

$$\vec{F} + \underbrace{m\frac{d\vec{v}}{dt}}_{inertial\ force} + \underbrace{m\frac{d\vec{\omega}}{dt}\times r}_{Euler's\ force} + \underbrace{2m\vec{\omega}\times\frac{d\vec{r}}{dt}}_{Coriolis\ force} + \underbrace{m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}\right)}_{centrifugal\ force} = 0 \quad (1)$$

Application of this general equation requires specification of individual forces according to actual conditions and/or eventually implementing other acting forces. We will consider forces originated by motion of real body – induced with resistances (losses) in addition to curvilinear motion forces.

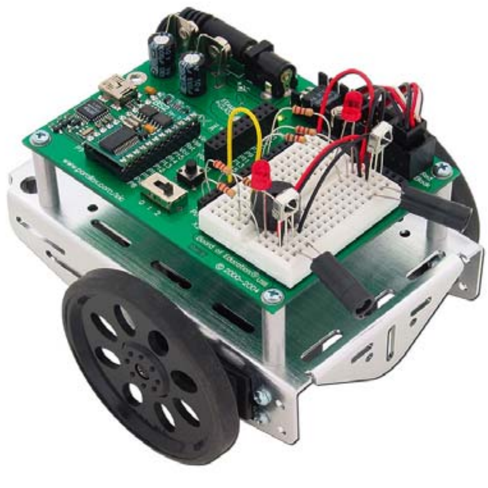


Fig. 1 Differentially steered mobile robot

We will approximate these forces in simplest manner to be proportional to a speed. Equations describing dependences of translation and rotation speed of selected chassis point to actual wheel motor voltages will be result of the dynamic part.

Selection of the point where actual translation and rotation speed will be evaluated influences significantly initial equations and hence complexity of the resulting model. If the selected point is centre of gravity then initial equations of dynamic part are simplest but equations describing dependencies between wheel speeds and translation and rotation speed are more complicated. Centre of the join between wheels is used as the selected chassis point in common literature. Such a choice leads to simplest recalculation of actual wheel speeds to motion equations of that point. Trade-off between these two approaches is chosen in our paper – point as centre of gravity projection to join between wheels is selected. Trajectory (time course) computation of another chassis points (points where wheels meet the ground) supplements dynamic part of the model.

2. MATHEMATICAL MODEL

Described mobile robot is driven by two DC motors with common voltage source and independent control of each motor. Motors are connected with driving wheels through gear-box with constant gear ratio. Ideal gear-box means that it reduces linearly angular speed and boosts the moment (nonlinearities are not considered). Loses in motor and also in gear-box are proportional to rotation speed. Chassis is equipped with caster wheel with no influence to chassis motion (its influence is included in resistance coefficients acting against motion).

Model of the robot consists from three relatively independent parts. Description of the ideal DC motors connected in series is given in chapter 2.1. Two equations describe dependency of the motor rotation speed and current on power supply voltage and loading moment related to chassis dynamics. Motion equations are presented in chapter 2.2 – dependency between translation and rotation speed of the selected point on moments acting to driving wheels. Chapter 2.3 is dedicated to equations describing how motor speed influences translation and rotation speed of the selected point and to complete model formulation. In last chapter 2.4 the model is transformed to simpler form which is more suitable for next using and for trajectory of arbitrary point calculation. Equations describing trajectory corresponding to contact points of the driving and caster wheels with the ground are formulated.

2.1 DC motor in series connection dynamics

Equivalent circuit of ideal DC motor in series connection (Poliak et al. 1987) is in Fig. 2. It consists from resistivity R, inductance L and magnetic field of the motor *M*. Commutator is not considered. Rotor produces electrical voltage with reverse polarity than source voltage – electromotive voltage U_M, which is proportional to rotor angular velocity ω. Twisting moment of the rotor M_M is proportional to current *i*.

Ideal behaviour means that whole electric energy used to magnetic field creating is transformed without any loses to

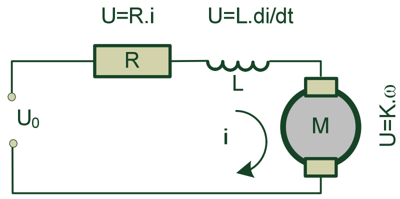


Fig. 2 Equivalent circuit of motor

mechanical energy – moment of the motor. We do not consider loses in magnetic field but only electric loses in winding and mechanical loses proportional to rotor speed.

Firs equation describes motor behaviour through balancing of voltages (Kirhoff's laws)

$$U_R + U_L = U_0 - U_M\,, \quad Ri + L\frac{di}{dt} = U_0 - K\omega \qquad (2)$$

where

R	[Ω]	is motor winding resistivity,
L	[H]	is motor inductance,
K	[kg.m^2.s^{-2}.A^{-1}]	is electromotoric constant,
U_0	[V]	is source voltage,
ω	[rad.s^{-1}]	is rotor angular velocity and
i	[A]	is current flowing through winding.

Second equation is balance of moments (electric energy) – moment of inertia M_s, rotation resistance proportional to rotation speed (mechanical loses) M_o, load moment of the motor M_x and moment M_M caused by magnetic field which is proportional to current

$$\begin{aligned} M_s + M_o + M_x &= M_M \\ J\frac{d\omega}{dt} + k_r\omega + M_x &= Ki \end{aligned} \qquad (3)$$

where

J	[kg.m^2]	is moment of inertia,
k_r	[kg.m^2.s^{-1}]	is coefficient of rotation resistance and
M_x	[kg.m^2.s^{-2}]	is load moment.

2.2 Chassis dynamics

Chassis dynamics is defined with vector of translation speed v_B acting in selected chassis point and with rotation of this vector with angular velocity ω_B (constant for all chassis points). It is possible to calculate trajectory of arbitrary chassis point from these variables. Point B for which the equations are derived is centre of gravity normal projection to join between wheels – see Fig. 3. This leads according to authors to simplest set of equation for whole model. We consider general centre of gravity T position – usually it is placed to centre of the join between wheels.

We consider forces balances as starting equations. It is possible to replace two forces F_L and F_P acting to chassis in left (L) and right (P) wheel ground contact points with one force F_B and torsion moment M_B acting in point B. Chassis is characterized with semi-diameter of the driving wheels *r*, total weight *m*, moment of inertia J_T with respect to centre of gravity located with parameters l_T, l_L, l_P.

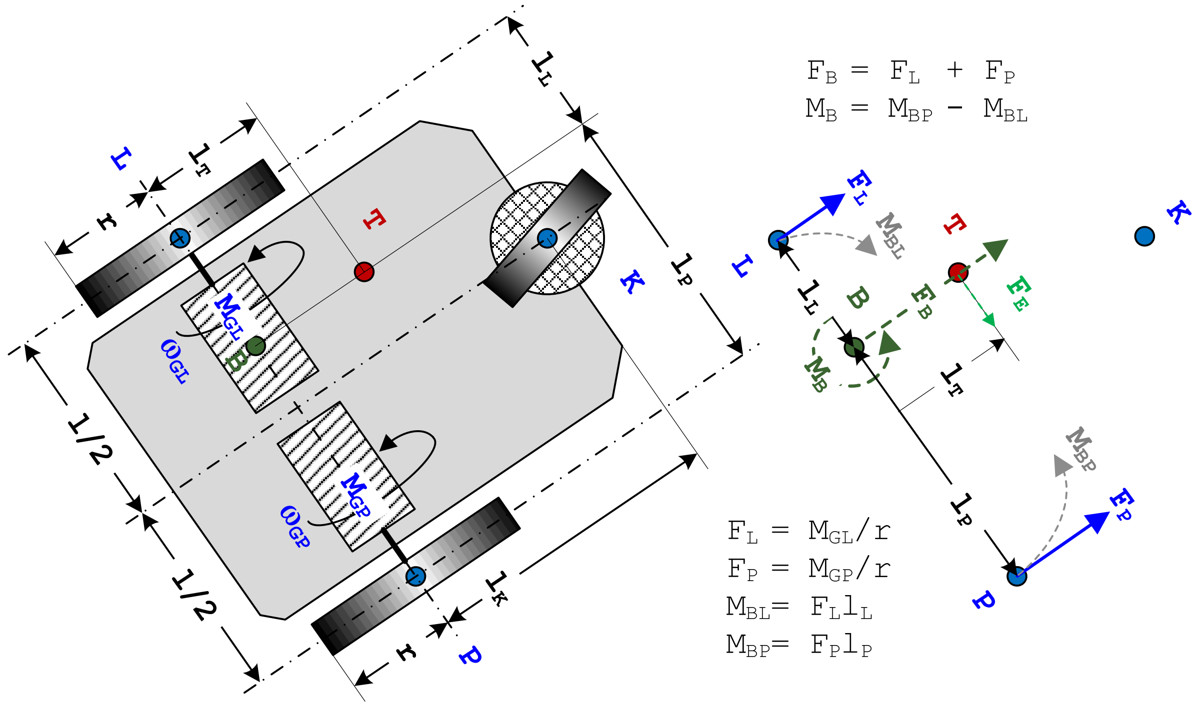


Fig. 3 Chassis scheme and forces

Let us specify equation (1) for our case. Position of the centre of gravity is constant with respect to axis of rotation so we do not need to consider Coriolis force. We have to consider Coriolis force for example if the chassis moves on rotating surface.

Similarly we do not consider centrifugal force – chassis is supposed to be solid body represented as mass point (centre of gravity). Because force vector causing the movement acts in point B and goes through centre of gravity it is enough if we consider inertial force by linear motion. By rotational motion it is necessary to consider moment caused with Euler's force because the axis of rotation does not go through the centre of gravity.

By the balance of forces causing linear motion we will consider except of forces F_L, F_P caused by drives and inertial force F_S also resistance force F_O proportional to speed v_B. The balance of forces influencing linear motion is

$$\begin{gathered} F_L + F_P + F_O + F_S = 0 \\ \frac{M_{GL}}{r} + \frac{M_{GP}}{r} - k_v v_B - m\frac{dv_B}{dt} = 0 \end{gathered} \tag{4}$$

where

m [kg] is robot mass,
k_v [kg.s^{-1}] is resistance coefficient against linear motion
M_{GL} [kg.m^2.s^{-2}] is moment of the left drive,
M_{GP} [kg.m^2.s^{-2}] is moment of the right drive,
v_B [m.s^{-1}] is linear motion speed and
r [m] is semi-diameter of the wheels.

Balance of moments is slightly more complicated because the rotation axis does not lie in centre of gravity. That's why it is necessary to take into account except chassis momentum M_T also moment $M_E = l_T F_E$ caused by Euler's force F_E. Similarly as by linear motion we will consider moment M_O caused with resistance against rotation to be proportional to angular velocity ω_B.

$$\begin{gathered} M_{BL} + M_{BP} + M_O + M_T + M_E = 0 \\ -\frac{M_{GL}}{r} l_L + \frac{M_{GP}}{r} l_P - k_\omega \omega_B - J_T \frac{d\omega_B}{dt} - l_T m \frac{d\omega_B}{dt} l_T = 0 \end{gathered} \tag{5}$$

where

l_P [m] is distance of the right wheel from point B,
l_L [m] is distance of the left wheel from point B,
l_T [m] is distance of the centre of gravity from point B,
k_ω [kg.m^2.s^{-1}] is resistance coefficient against rotary motion
J_T [kg.m^2] is moment of inertia with respect to rotation axis in centre of gravity and
ω_B [s^{-1}] is angular speed in point B.

Resulting moment of inertia J_B with respect to rotation axis in point B is given by eq. (6) which is parallel axis theorem or Huygens-Steiner theorem - see e.g. (Horák et al. 1976)

$$J_B = J_T + m l_T^2 \tag{6}$$

where

J_T [kg.m^2] is moment of inertia with respect to centre of gravity and
l_T [m] is distance between centre of gravity and point B.

2.3 Relationship between rotation speed of the motor and centre of gravity chassis movement (kinematics)

The equation describing the behaviour of the two motors (currents and angular velocity) and the behaviour of the chassis (the speed of the linear movement and speed of the rotation) are connected only through moments of engines.

Equations express the law of conservation of energy which is conversion of electrical energy to mechanical including one type of losses but represent only one relationship between the speed of the two motors (peripheral speed of the drive wheels) and rates of movement and rotation of the chassis. Additional relation is given by design of the drive and chassis. We expect that both drive wheels are firmly linked to rotors of relevant engines over ideal gearbox with gear ratio p_G - without nonlinearities and any flexible members.

Gearbox decreases output angular velocity ω_{Gx} with relation to the input angular speed ω_x according to the transmission ratio p_G and simultaneously in the same proportion increases output torque M_{Gx} with relation to the input torque M_x.

$$\omega_{GL} = \frac{\omega_L}{p_G} \qquad \omega_{GP} = \frac{\omega_P}{p_G} \tag{7a}$$

$$M_{GL} = p_G M_L \qquad M_{GP} = p_G M_P \tag{7b}$$

Further we assume that both drive wheels have the same radius r and their peripheral speeds v_L, v_P depend on the angle speeds of gearbox output ω_{GL}, ω_{GP} according to relations

$$\begin{aligned} v_L &= r\omega_{GL} = r\frac{\omega_L}{p_G} \\ v_P &= r\omega_{GP} = r\frac{\omega_P}{p_G} \end{aligned} \tag{7c}$$

To determine the value of the linear speed in point B and the angular velocity of rotation let us start from Figure 4. We expect that both drive wheels have the same axis of rotation and therefore their peripheral speeds are always parallel. The illustration shows the positioning where the peripheral speeds v_L and v_P actually operate (driving wheels L and P) and the point B. We want to specify such a linear v_B and angular ω_B speeds that have the same effect as the action of the peripheral speed of the driving wheels. By using the similarity of triangles depicted in Figure 4 we can recalculate the peripheral speeds of the wheels v_L, v_P to the speed v_B in point B according to relation (8a) and the angular velocity of rotation ω_B according to the relation (8b)

$$v_B = \frac{v_L l_P + v_P l_L}{l_L + l_P} = \frac{r}{p_G(l_L + l_P)}(l_P\omega_L + l_L\omega_P) \tag{8a}$$

$$\omega_B = \frac{v_B}{x + l_L} = \frac{v_P - v_L}{l_L + l_P} = \frac{r}{p_G(l_L + l_P)}(-\omega_L + \omega_P) \tag{8b}$$

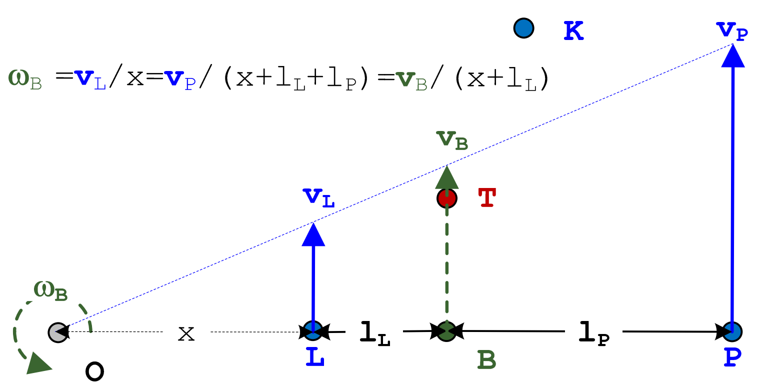


Fig. 4 Linear and angular speeds recalculations

2.4 Trajectory calculation of the arbitrary chassis points

We can determine from linear speed of v_B and angular speed ω_B (motion equations) current rotation angle α of the chassis and the current position (the coordinates x_B, y_B) of point B (Šrejtr 1954) according to relations

$$\frac{d\alpha}{dt} = \omega_B \tag{9a}$$

$$\frac{dx_B}{dt} = v_B\cos(\alpha) \tag{9b}$$

$$\frac{dy_B}{dt} = v_B\sin(\alpha) \tag{9c}$$

To determine the current position of the contacts of all three chassis wheels (points L, P and K) with ground we need to know the location of these points in relation to point B. This location is shown in Figure 5. From geometric dimensions we determine equation describing the relative position of these points in relation to point B depending on the angle of rotation.

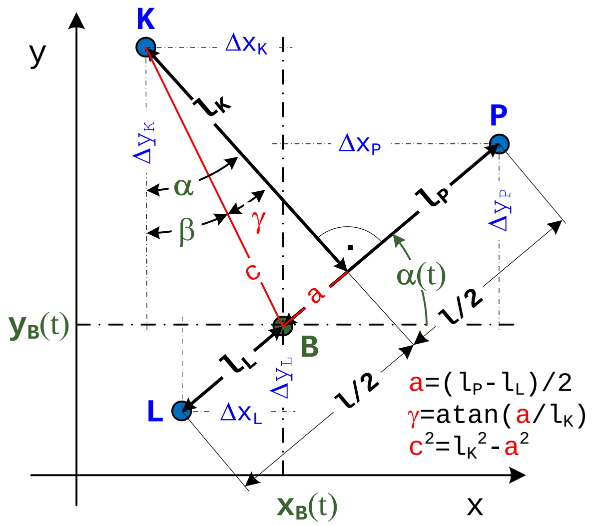


Fig. 5 Arbitrary chassis point recalculation

Relative positions Δx_L, Δy_L of the point L and Δx_P, Δy_P of the point P depending on angle of rotation α are given by

$$\Delta x_L = -l_L\sin(\alpha) \qquad \Delta y_L = -l_L\cos(\alpha) \tag{10a}$$

$$\Delta x_P = +l_P\sin(\alpha) \qquad \Delta y_P = +l_P\cos(\alpha) \tag{10b}$$

To determine the relative position Δx_K, Δy_K of the point K we use an auxiliary right triangle specified by cathetus c and hypotenuses a and l_K (see Figure 5). Then the equations for relative coordinates of the point K calculating are

$$\begin{aligned} a &= \tfrac{1}{2}(l_P - l_L) \\ \gamma &= \arctan\left(a/l_K\right) \qquad c = \sqrt{a^2 + l_K^2} \\ \Delta x_K &= -c\sin(\alpha - \gamma) \qquad \Delta y_K = c\cos(\alpha - \gamma) \end{aligned} \tag{10c}$$

2.5 Overall model and steady-state

The dynamic part of the model consists from four differential equations describing the behaviour of both motors, two differential equations describing the dynamics of the chassis and two algebraic equations with dependency of linear and angular chassis speed on the peripheral speeds of the driving wheels. We can find in these equations eight state variables describing the current state of the left motor (current i_L, angular velocity of the rotor ω_L, loading moment M_L) and the right motor (current i_P, angular velocity of the rotor ω_P, loading moment M_P) and the movement of the chassis (linear speed v_B and angular velocity of rotation ω_B). All the state

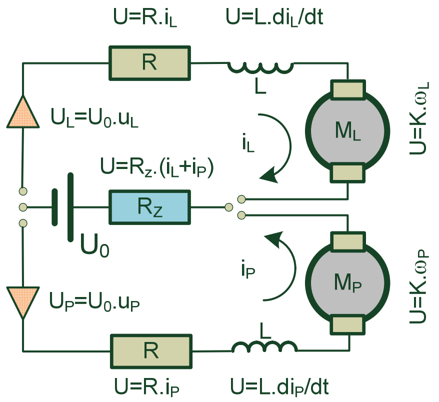


Fig. 6 Motors wiring

variables are dependent on the time courses of the power of the left U_L and right U_P motor.

Each motor has its own power supply voltage (U_L, U_P) disbranched from the common source of voltage U_0. Control of the supply voltage of both motors using amplifier with control signal u_x is shown in Figure 6. Because both engines are powered from the common source it will be taken into account also effect of the internal resistance R_z. Both motors are considered with the same parameters. We can write with using the equations (2) and (3) and Figure 6 four differential equations describing the behaviour of both engines as

$$Ri_L + R_z(i_L + i_P) + L\frac{di_L}{dt} = u_L U_0 - K\omega_L \quad (11a)$$

$$Ri_P + R_z(i_L + i_P) + L\frac{di_P}{dt} = u_P U_0 - K\omega_P \quad (11b)$$

$$J\frac{d\omega_L}{dt} + k_r\omega_L + M_L = Ki_L \quad (12a)$$

$$J\frac{d\omega_P}{dt} + k_r\omega_P + M_P = Ki_P \quad (12b)$$

Differential equations (4) and (5) describing the behaviour of the chassis complete the dynamic model. We can rewrite these equations with respect to the equations (7) and introduction of the "reduced" radius of the wheel r_G and total moment of inertia J_B (13a) as

$$r_G = \frac{r}{p_G} \qquad J_B = J_T + ml_T^2 \quad (13a)$$

$$\frac{p_G}{r}M_L + \frac{p_G}{r}M_P - k_v v_B - m\frac{dv_B}{dt} = 0$$
$$M_L + M_P - r_G k_v v_B - r_G m\frac{dv_B}{dt} = 0 \quad (13b)$$

$$-l_L\frac{p_G}{r}M_L + l_P\frac{p_G}{r}M_P - k_\omega\omega_B - \left(J_T + ml_T^2\right)\frac{d\omega_B}{dt} = 0$$
$$-l_L M_L + l_P M_P - k_\omega\omega_B - r_G J_B\frac{d\omega_B}{dt} = 0 \quad (13c)$$

It is possible to rewrite the last two algebraic equations (8a) a (8b) describing the dependence between rotations speed of both motors and chassis movement with using the substitution (13a) as

$$v_B = \frac{r_G}{l_L + l_P}\left(l_P\omega_L + l_L\omega_P\right) \quad (14a)$$

$$\omega_B = \frac{r_G}{l_L + l_P}\left(-\omega_L + \omega_P\right) \quad (14b)$$

These six differential equations (11a,b), (12a,b), (13b,c) and two algebraic equations (14a,b) containing eight state variables representing a mathematical description of dynamic behaviour of ideal differentially steered mobile robot with losses linearly dependent on the revolutions or speed. Control signals u_L and u_P that control the supply voltages of the motors are input variables and the speed of the movement v_B and speed of rotation ω_B are output variables. From them with using the equations (9a) – (9c) we can determine the current coordinates of a point B and the angle of rotation of the chassis.

In the following calculation of steady-state values for constant engine power voltages is given. Calculation of steady-state is useful both for the checking of derived equations and secondly for the experimental determination of the values of the unknown parameters. Because the equation (11)-(14) are linear with respect to state variables the calculation of steady-state leads to a system of eight linear equations which we can write in matrix form as

$$\begin{bmatrix} R+R_z & R_z & K & 0 & 0 & 0 & 0 & 0 \\ R_z & R+R_z & 0 & K & 0 & 0 & 0 & 0 \\ K & 0 & -k_r & 0 & -1 & 0 & 0 & 0 \\ 0 & K & 0 & -k_r & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -r_G k_v & 0 \\ 0 & 0 & 0 & 0 & -l_L & l_P & 0 & -r_G k_\omega \\ 0 & 0 & l_P & l_L & 0 & 0 & -\frac{l_P + l_L}{r_G} & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -\frac{l_P + l_L}{r_G} \end{bmatrix} \begin{bmatrix} i_L \\ i_P \\ \omega_L \\ \omega_P \\ M_L \\ M_P \\ v_B \\ \omega_B \end{bmatrix} = \begin{bmatrix} U_L \\ U_P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (15)$$

2.6 Computational form of the model

A mathematical model will be used in particular for the design and simulation validation of control movement of the robot. Model can be divided into three series-involved parts as shown in Figure 7. From the control point of view action variables are signals u_L a u_P that control the supply voltage of the motors. Momentary speed v_B and speed of rotation ω_B are output variables from linear part of the model. These variables are the inputs to the consequential non-linear part of the model, whose outputs are controlled variables - the coordinates of selected point position x_B, y_B and the rotation angle of the chassis α. The last part is the calculation of coordinates of the position of arbitrary points of the chassis.

We can modify linear part of the model into simpler form for control design purposes – to reduce number of differential equations from six to four. If we substitute equations (14a,b) into (13b,c) and eliminate moments M_L a M_P by substitution

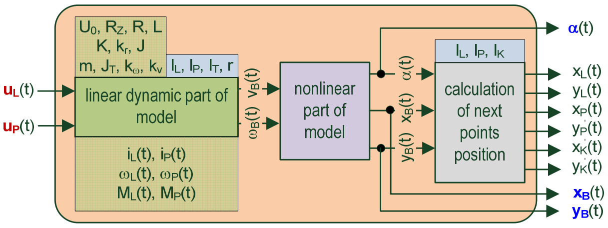


Fig. 7 Model partitioning into linear and nonlinear part

of (12a,b) to (13b,c) we are able to reduce four differential equations (12a,b) a (13b,c) into two (17c,d).

If we introduce substitution of the parameters according to following formulas

$$a_L = k_r + \frac{k_v l_P r_G^2}{l_L + l_P} \qquad a_P = k_r + \frac{k_v l_L r_G^2}{l_L + l_P} \tag{16a}$$

$$b_L = J + \frac{m l_P r_G^2}{l_L + l_P} \qquad b_P = J + \frac{m l_L r_G^2}{l_L + l_P} \tag{16b}$$

$$c_L = k_r l_L + \frac{k_\omega r_G^2}{l_L + l_P} \qquad c_P = k_r l_P + \frac{k_\omega r_G^2}{l_L + l_P} \tag{16c}$$

$$d_L = J l_L + \frac{J_B r_G^2}{l_L + l_P} \qquad d_P = J l_P + \frac{J_B r_G^2}{l_L + l_P} \tag{16d}$$

The reduced linear part of the model consists from set of equations

$$\frac{di_L}{dt} = \frac{u_L U_0 - K.\omega_L - (R + R_z).i_L - R_z i_P}{L} \tag{17a}$$

$$\frac{di_P}{dt} = \frac{u_P U_0 - K.\omega_P - (R + R_z).i_P - R_z i_L}{L} \tag{17b}$$

$$\frac{d\omega_L}{dt} = \frac{1}{b_L d_P + b_P d_L}\left(d_P\left[K(i_L + i_P) - a_L\omega_L - a_P\omega_P\right] - b_P\left[K(-l_L i_L + l_P i_P) + c_L\omega_L - c_P\omega_P\right]\right) \tag{17c}$$

$$\frac{d\omega_P}{dt} = \frac{1}{b_L d_P + b_P d_L}\left(d_L\left[K(i_L + i_P) - a_L\omega_L - a_P\omega_P\right] + b_L\left[K(-l_L i_L + l_P i_P) + c_L\omega_L - c_P\omega_P\right]\right) \tag{17d}$$

and output variables are given by algebraic equations (14a,b). It is possible to write reduced linear part of the model as standard state-space model in matrix form as

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax} + \mathbf{Bu} \qquad \mathbf{x} = \begin{bmatrix} i_L \\ i_P \\ \omega_L \\ \omega_P \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_L \\ u_P \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} v_B \\ \omega_B \end{bmatrix} \tag{18a}$$
$$\mathbf{y} = \mathbf{Cx}$$

with constant matrices **A**, **B** a **C**

$$\mathbf{A} = \begin{bmatrix} -\frac{R+R_z}{L} & -\frac{R_z}{L} & -\frac{K}{L} & 0 \\ -\frac{R_z}{L} & -\frac{R+R_z}{L} & 0 & -\frac{K}{L} \\ \frac{K(d_P + b_P l_L)}{b_L d_P + b_P d_L} & \frac{K(d_P - b_P l_P)}{b_L d_P + b_P d_L} & -\frac{d_P a_L + b_P c_L}{b_L d_P + b_P d_L} & -\frac{d_P a_P - b_P c_P}{b_L d_P + b_P d_L} \\ \frac{K(d_L - b_L l_L)}{b_L d_P + b_P d_L} & \frac{K(d_L + b_L l_P)}{b_L d_P + b_P d_L} & -\frac{d_L a_L - b_L c_L}{b_L d_P + b_P d_L} & -\frac{d_L a_P + b_L c_P}{b_L d_P + b_P d_L} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} \frac{U_0}{L} & 0 \\ 0 & \frac{U_0}{L} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \frac{l_P r_G}{l_L + l_P} & \frac{l_L r_G}{l_L + l_P} \\ 0 & 0 & -\frac{r_G}{l_L + l_P} & \frac{r_G}{l_L + l_P} \end{bmatrix}$$

3. EXAMPLE OF THE BEHAVIOUR

Basic verification of the above derived model was made by calculation for situations where we can guess the behaviour of the real device. First value of the state variables in steady states will be given for some combinations of parameters and motor supply voltages. Further time courses of the robot trajectory will be determined for some combinations of time courses of supply voltages when robot is starting from zero speed.

Values of the parameters listed in the following tables are used in all of the calculations. These values are chosen so that they at least roughly correspond to the values estimated for the robot in Figure 1. The values of the geometrical and other parameters of the chassis are listed in Table 1.

Table 1 Chassis parameters

Notation	Value	Dimension	Meaning
l_L	0.040	m	distance of the left wheel from point B
l_P	0.060	m	distance of the right wheel from point B
l_T	0.020	m	distance of centre of gravity from join between wheels
l_K	0.040	m	distance of caster wheel from join between wheels
r	0.050	m	semi-diameter of driving wheel
m	1.250	kg	total weight of the robot
k_v	0.100	$kg.s^{-1}$	coefficient of the resistance against robot linear motion
J_T	0.550	$kg.m^2$	moment of inertia of robot with respect to centre of gravity
k_ω	1.350	$kg.m^2.s^{-1}$	coefficient of the resistance against robot rotating

Necessary parameters for DC motors with common voltage source description are given in Table 2. We consider identical motors with identical parameters.

Table 2 DC motors parameters

Notation	Value	Dimension	Meaning
R	2.000	Ω	motor winding resistivity
L	0.050	H	motor inductance
K	0.100	$kg.m^2.s^{-2}.A^{-1}$	electromotoric constant
R_Z	0.200	Ω	source resistance
U_0	10.00	V	source voltage
J	0.025	$kg.m^2$	total moment of inertia of rotor and gearbox
k_r	0.002	$kg.m^2.s^{-1}$	coefficient of the resistance against rotating of rotor and gearbox
p_G	25	---	gearbox transmission ratio

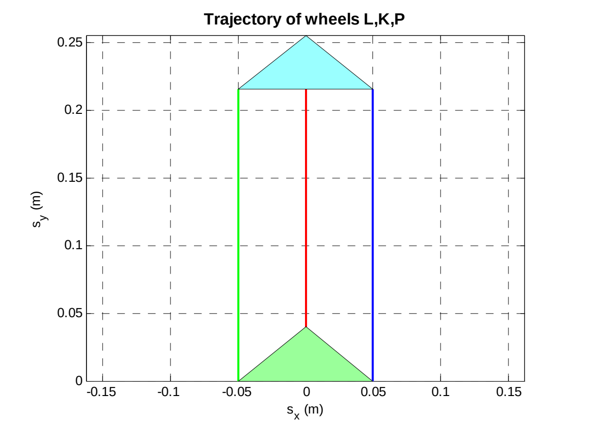
3.1 Steady state for different positions of point B and motors voltages

The steady states are calculated as a solution of the system of eight equations in matrix form (15). Traces of the wheels are shown during the first 20 seconds of motion from zero initial conditions - calculated from state-space model (18) and from the equations for the trajectories calculation (9,10). Trajectories are plotted for the situation that the origin of the coordinate system is in the centre between the wheels, which is on the x-axis and the default orientation of the robot is in the direction of the y axis. Starting and final position of the robot is displayed using the triangle that connects all three wheels. Trajectory of the centre of gravity is displayed in addition to the traces of the wheels.

Steady-state A (Table 3a) corresponds to the geometric arrangement - point (B) is midway between the wheels and both engines have the same supply voltage. The result is that the robot moves only linearly.

Table 3a Steady state A

	left wheel	right wheel	
U	1.000	1.000	V
l	0.050	0.050	m
i	0.13514	0.13514	A
ω	1.07534	1.07534	Hz
M	0.000001	0.000001	N.m
v_B	0.0013513		$m.s^{-1}$
ω_B	0.0000000		Hz



The following three experiments show the influence of centre of gravity position. Steady-state B (table 3b) holds again for the symmetric geometric arrangement but only one motor is powered. Steady state C (table 3c) shows the situation in the case that point B is in the extreme position above the left wheel and only the left motor is powered. Steady-state D (table 3d) corresponds to the same position of the point B above the left wheel but is only right motor is powered. In all three cases the robot rotates and at the same time the point B has some linear speed. Both wheels produce translation because of the interactions.

Table 3b Steady state B

	left wheel	right wheel	
U	0.000	1.000	V
l	0.050	0.050	m
i	-0.02772	0.16287	A
ω	0.04523	1.07935	Hz
M	-0.001176	1.03010	N.m
v_B	0.006757		$m.s^{-1}$
ω_B	0.123762		Hz

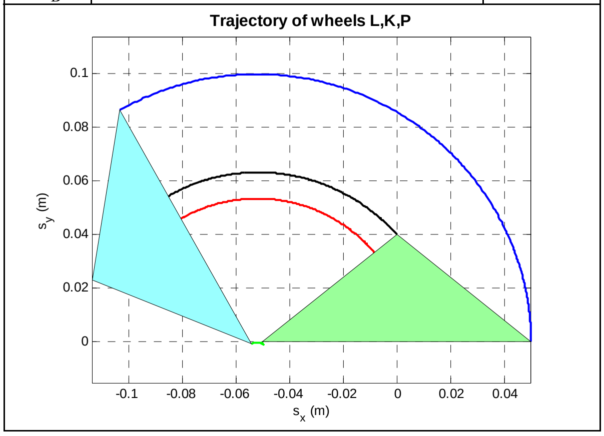


Table 3c Steady state C

	left wheel	right wheel	
U	1.000	0.000	V
l	0.000	0.100	m
i	0.16288	-0.02773	A
ω	1.030062	0.045243	Hz
M	0.00334400	-0.00334141	N.m
v_B	0.0129441		$m.s^{-1}$
ω_B	-0.1237559		Hz

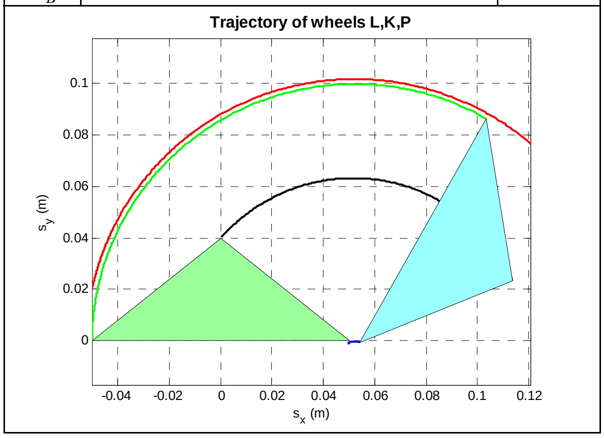
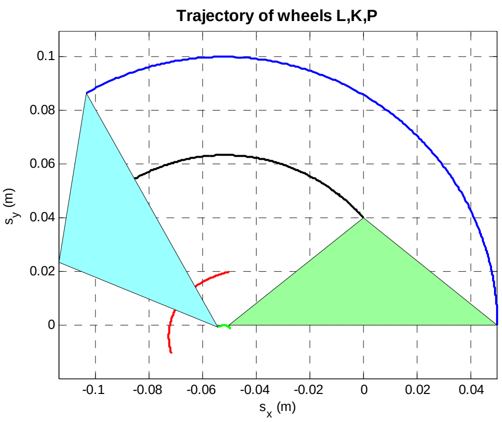


Table 3d Steady state D

	left wheel	right wheel	
U	0.000	1.000	V
l	0.000	0.100	m
i	-0.02773	0.16286	A
ω	0.045248	1.030120	Hz
M	-0.00334148	0.00334159	N.m
v_B	0.0005686		m.s^{-1}
ω_B	0.1237626		Hz



3.2 Dynamic behaviour for particular cases

Dynamic behaviour is demonstrated on the time courses of currents and angular speeds of the motors starting from zero initial conditions. Graphs in Figure 8 show courses of supply voltages, currents and angular speeds for the case that the point B is in the middle between both motors with the same constant voltage 1 V. Situation corresponds to experiment with the parameters in Table 3a.

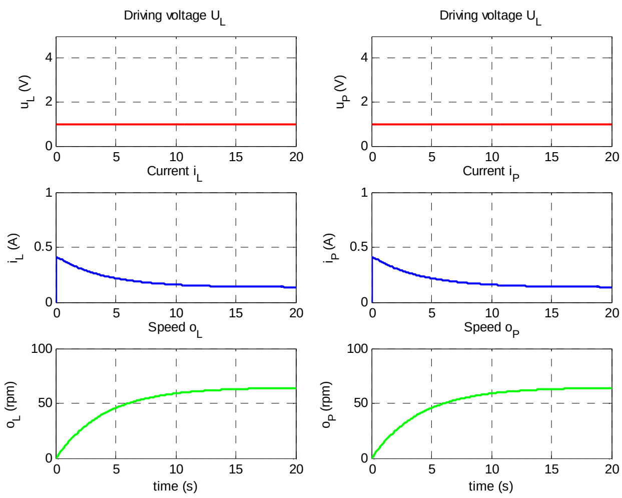


Fig. 8 Dynamic behaviour - constant supply voltage 1 V for both motors

Situation where point B is in the middle between both motors with the right motor voltage 1 V only corresponds to experiment with the parameters in Table 3b.

Illustrative example of behaviour in the situation when both voltages are periodic and with different amplitudes is in figures 10 and 11. On the left motor is a rectangular voltage of period 20 s, phase offset 10 s and amplitude 3. On the right motor is a rectangular voltage of doubled period 40 s and amplitude 4.

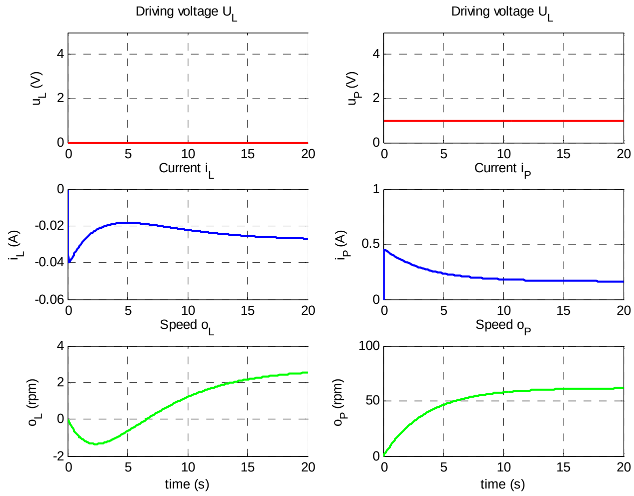


Fig. 9 Dynamic behaviour - constant supply voltage 1 V for right motor

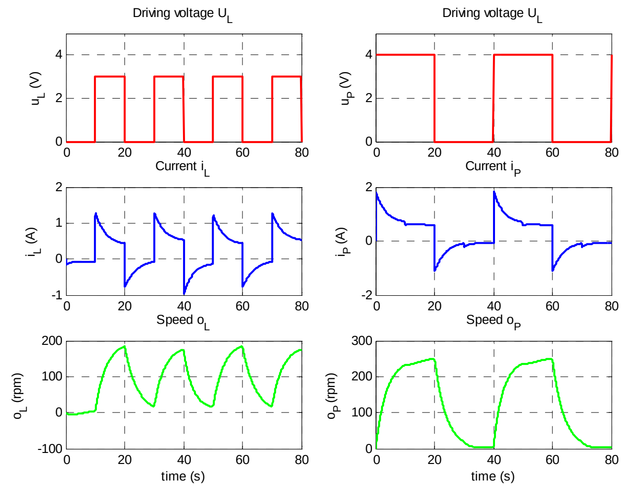


Fig. 10 Dynamic behaviour - periodic voltages

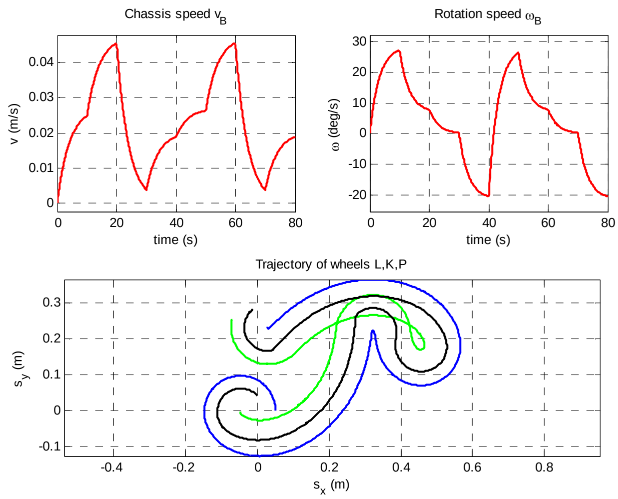


Fig. 11 Dynamic behaviour - periodic voltages – speeds and trajectories

4. CONCLUSION

The behaviour of the dynamic model in simulated situations agrees with the expected behaviour of a real device. Position of centre of gravity does not affect the behaviour in steady state. Immediate linear speed in point B depends on its position but the trajectories of the wheels are independent on the position of the point B.

Interaction of the two drives was confirmed. Because of the forces of inertia and the forces of resistance also wheel without supply voltage rotates by the chassis movement. Even change of the meaning of the rotation occurs in the transient state. This situation is seen in Figure 9.

Motor dynamics is negligible compared to the expected dynamics of the chassis for estimated motor parameters. Because the parameters of the model have physical meaning it will be possible to measure directly some parameters on real device. Identification of additional parameters will be possible experimentally from measured time courses of power voltages and the corresponding courses of angular speed of the wheels.

ACKNOWLEDGMENTS

The work has been supported in the framework of the research project MSM 0021627505 in part "Management, optimization and diagnostics of complex systems". This support is very gratefully acknowledged.

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